Theoretische Physik III: Klassische Elektrodynamik, Exercise 2

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The Yukawa potential is given by the expression

$$\phi_Y(\mathbf{r}) = \frac{e^{-mr}}{r} \tag{1}$$

where $r = |\mathbf{r}|$ and $m > 0$ is the mass of the particle that mediates the potential. The inverse mass is a length scale (Compton length) that determines the range of the potential. If photons had a rest mass, the Coulomb potential would have to be replaced by the Yukawa potential. We can see that the Coulomb potential is the limiting case of $\phi_Y(\mathbf{r})$ in the zero-mass limit (infinite-range limit).

1. Green's function for the Poisson equation (Oral)

In this exercise, we are going to prove the important identity

$$\Delta \frac{1}{|\mathbf{r}|} = -4\pi\, \delta^3(\mathbf{r}) \tag{2}$$

in two different ways.

a) Show that $\Delta \frac{1}{|\mathbf{r}|} = 0$ for any $\mathbf{r} \neq 0$. Then, use Gauss's law to prove (2) by integrating over a sphere.

b) Use the three-dimensional Fourier transformation of $\Delta \frac{1}{|\mathbf{r}|}$ to show that equation (2) holds. For this purpose, first calculate the Fourier transform of $\frac{1}{|\mathbf{r}|}$. It is useful to do this by transforming the Yukawa potential and taking the zero-mass limit afterwards.

Now, using the result from part (b),

c) show that the Yukawa potential fulfills

$$\left[\Delta - m^2\right]\phi_Y(\mathbf{r}) = -4\pi\, \delta^3(\mathbf{r}). \tag{3}$$

2. Cavendish experiment, part 1: Spherical capacitor (Oral)

A spherical capacitor is given by two concentric, metallic spherical shells with radii $R_1 < R_2$ and respective charges $Q_1 = Q$ and $Q_2 = -Q$.

a) Calculate the electric field $\mathbf{E}(\mathbf{r})$ of the given arrangement for the three regions (i) $r < R_1$, (ii) $R_1 < r < R_2$ and (iii) $R_2 < r$.

b) Determine the scalar potential $\phi(\mathbf{r})$ with the boundary condition $\phi(\mathbf{r}) \longrightarrow 0$ for $r \longrightarrow \infty$ such that it is continuous at R_1 and R_2.

c) Determine the capacitance of the spherical capacitor.

3. Cavendish experiment, part 2 (Written) [4pt]

In 1772, Cavendish designed the following experiment to verify Coulombs law. He used a spherical capacitor with the two spheres initially connected by an electrical contact. He then placed a static charge on the outer sphere and subsequently removed the contact between the spheres. After removing the outer sphere, he confirmed that the inner sphere had not been charged.

This is a special property of the $1/r$ Coulomb law where the electric potential inside a conducting sphere is constant (the electric field vanishes). In the following, we are going to assume that the electrostatic potential is instead given by

a) the modified power law $\phi_\epsilon(\mathbf{r}) = \frac{1}{|\mathbf{r}|^{1-\epsilon}}$,

b) the Yukawa potential with a finite photon mass.

Determine the charge on the inner sphere for both prospective potentials as follows. Let V_i be the "volume" of the ith sphere. We can calculate the electrostatic interaction energy by

$$E_{ij} = \frac{1}{2} \int_{V_i} \mathrm{d}^3 r \int_{V_j} \mathrm{d}^3 r' \, \rho(\mathbf{r}) \rho(\mathbf{r}') \phi(\mathbf{r} - \mathbf{r}'), \tag{4}$$

where E_{ii} is the self energy of the ith sphere and $E_{12} + E_{21} = 2E_{12}$ is the interaction energy between the two spheres. The full energy is given by $E = E_{11} + E_{22} + 2E_{12}$.

i) Due to the spherical symmetry, the charge density will be homogeneously distributed on the spheres. Assume that the spheres have a surface density of $\sigma_i = Q_i/S_i$ were S_i is the surface area of the respective sphere. Calculate the total energy $E(Q_1, Q_2)$ for arbitrary charges.

ii) Determine the actual charge distribution by minimizing the energy with the constraint that the total charge $Q = Q_1 + Q_2$ is fixed.

Remark (Bonus): From the energy function $E(Q_1, Q_2)$ for the Coulomb potential (set $\epsilon = 0$ or $m = 0$ in your solution) we can derive the capacitance from exercise 2c. How can this be done?