Theoretische Physik III: Klassische Elektrodynamik, Exercise 3

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1. Method of images, Part 1 (Oral)

Consider a conducting plane (yz-plane at $x = 0$) and a conducting sphere of radius $R \ll L$ with the center at $(L, 0, 0)$. A point charge q_1 is located between the plane and the sphere at $(r, 0, 0)$; $0 < r < L - R$. Your goal is to determine the electrostatic potential using *image charges* in lowest order in R/L. In order, to satisfy boundary conditions on two conducting objects it is necessary to construct complete set of mirror images.

Hint: Note that $L - r$ can be of the order of R.

a) First, find the image charge q_2 of q_1 which produces the correct boundary conditions on the $yz-$plane. Next, find image charges of q_1 and q_2 (call them q_3 and q_4, respectively) so that the boundary conditions on the conducting sphere are satisfied.

b) Determine image charges of q_3 and q_4 (call them q_5 and q_6, respectively).

c) All other image charges are of lower order in R/L. Take them into account assuming that there are two sets of charges at positions $x_\pm = \pm L$. For each position, sum up geometric series of image charges in the limit of $R \ll L$.

d) Determine the electrostatic potential.

2. Method of images, Part 2 (Written) [1 pt.]

Consider a conducting sphere of radius a in a uniform electric field E. Your goal is to determine the electrostatic potential generated by the sphere using the method of images.

The uniform field can be though of as being produced by appropriate positive and negative charges at infinity.

a) What is the relation between the charges ($\pm Q$) and distances ($\pm R$) which generate the electric field E in the center of the sphere?

b) Determine the electrostatic potential due to charges and their images for finite R.

c) Determine the limit $R \to \infty$ of the potential from part (b).

3. Conformal mapping (Oral)

Imagine that we know the complex potential $w(z)$ for a 2-dimensional electrostatic problem with point charges and conducting objects. Next, we change only the geometry of the problem by a conformal map f. If f^{-1} is also conformal in the region of our interest, then the potential of the new complex problem is given by $w(f^{-1}(z))$.

Solve the following problem using conformal mapping: A circular conductor with radius $R = 1$ at position $(0, 0)$ and the point charge q at position $\mathbf{r}_1 = (\xi < 1, 0)$ are given.

a) Calculate complex potential for a charge at position $\mathbf{r}_0 = (0, 0)$ inside circular conductor.

b) Find a conformal mapping of the form $f(z) = \frac{z+b}{cz+d}$ which maps the unit circle onto itself and $\mathbf{r}_0$ onto $\mathbf{r}_1$.

c) Determine complex and electrostatic potentials.

4. Complex potential (Written) [3 pt.]

An infinitesimally thin, conducting band has width $2a$ in the x-direction and is infinitely long in $z-$direction. It is located in an external, uniform $\mathbf{E}$-field parallel to the x-axis. Your goal will be to determine the complex potential $w(z)$ and electrostatic potential $\varphi(x, y)$. In contrary to the previous exercise where we used a conformal mapping to find the complex potential, here we will use symmetries of the system and the conformality of $w(z)$.

a) What conditions have to be satisfied by electric field components E_x, E_y (i) on the conducting band and (ii) for $x \to \pm\infty$? How do those conditions translated to the properties of $w(z)$?

b) The complex potential $w(z)$ has to be conformal on $\mathbb{C}\backslash[-a, a]$.

Show that

$$w(z) = b\sqrt{z-a}\sqrt{z+a} \tag{1}$$

satisfies the conformality condition. Find $b \in \mathbb{C}$ for which all conditions from part (b) are satisfied.

Hint: Recall that the square root function can be defined using $\log z$ as

$$\mathbb{C}\backslash\{0\} \in z \mapsto z^{1/2} := e^{1/2 \log z} \tag{2}$$

where we choose the branch cut to be $z \in [0, \infty[$. The complex logarithm can be expressed using real functions r, ϕ

$$\log(r(\cos\phi + i\sin\phi)) = \log r + i\phi, \qquad r \in \mathbb{R}_+, \quad \phi \in [0, 2\pi[\tag{3}$$

On the branch cut, the square root function $z^{1/2}$ is discontinuous. Intuitively, the jump of the argument of $z^{1/2}$ is equal to π.

c) Determine the electrostatic potential $\varphi(x, y)$ and plot its field lines and curves of constant potential.