Theoretische Physik III: Klassische Elektrodynamik, Exercise 4

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1. Cylindrical capacitor (Oral)

Consider two concentric conducting cylinders of length L with radii $R_1 < R_2$ (coaxial cable) that are separated from each other by an insulating medium such as air or vacuum. Compute the capacitance C per unit length for this setup for the case of very long conductors $L \gg R_2$. How does the capacitance change if the inner cylinder is not hollow, but solid instead?

2. Legendre polynomials and charge density (Written) [3 pts.]

Let a spherical shell with radius R carry a fixed charge density $\sigma(\theta)$ such that the potential on the sphere is

$$V(r = R, \theta) = V_0 + V_1 \cos\theta + V_2 \cos 2\theta \tag{1}$$

where V_0, V_1, and V_2 are constants and θ is the polar angle.

a) Find the potential $V(r, \theta)$ inside and outside of the spherical shell ($V(\infty) = 0$). Use the ansatz

$$\Phi(r, \theta) = \sum_{l=0}^{\infty} (a_l r^l + b_l r^{-(l+1)}) P_l(\cos\theta), \tag{2}$$

where $P_l(x)$ are the Legendre polynomials.

b) From the potential $V(r, \theta)$ compute the electric field $\mathbf{E}(r, \theta)$.

c) Oberserve how - as is to be expected - the component of $\mathbf{E}$ perpendicular to the spherical shell is discontinuous at $r = R$ with a jump of $\mathbf{E}_\perp^{\text{outside}} - \mathbf{E}_\perp^{\text{inside}} = \frac{\sigma}{\epsilon_0}$ while the tangential component is continuous. Use this to compute the surface charge distribution $\sigma(\theta)$.

Hint:
The first Legendre polynomials are given by

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = \frac{1}{2}(3x^2 - 1).$$

Furthermore they obey the orthogonality relation

$$\int_{-1}^{1} dx P_l(x) P_m(x) = \frac{2}{2l+1} \delta_{lm}. \tag{3}$$

3. Conducting tip (Written) [3 pts.]

The goal of this exercise is to work through pages 49 - 53 and 57 - 58 of the lecture notes to obtain a better understanding of the differential equations appearing in electrodynamics and their (physically meaningful) solutions. The previous exercise served as an introduction.

a) First, write the Laplace equation, $\nabla^2\Phi = 0$, in spherical coordinates and make an ansatz with separation of variables: $\Phi(\mathbf{r}) = u(r)P(\vartheta)\chi(\phi)/r$. Derive the differential Eq. (2.50) for $P(\vartheta)$ and regard it as an eigenvalue problem. In the lecture notes we take only solutions $l = 0, 1, 2, \ldots$ (Legendre polynomials $P_l(z), z \equiv \cos\vartheta$). Which solutions does one find for $l = -1, -2, \ldots$ and l non-integer ? How are they called ? When is one allowed to choose solutions with l non-integer ?

Consider now the grounded conical tip (or cut) of page 57 with angle $0 < \theta < \pi$. We are looking for the potential $\Phi(r, \vartheta, \phi)$ outside of the tip $(0 \leq \vartheta \leq \theta)$.

b) Starting from the Legendre differential Eq. (2.50), perform a change of variables $\xi \equiv \frac{1}{2}(1 - z)$ and derive Eq. (2.78) from it (now the separation constant l is called ν). When is one allowed to set $m = 0$? Solve the differential equation using a generalized power series ansatz

$$P(\xi) = \xi^\alpha \sum_{j=0}^{\infty} a_j \xi^j.$$

(This yields the expression immediately after Eq. (2.78).) For which ν do these solutions $P_\nu(z)$ have singularities on the interval $z \in [-1, 1]$? Do these singularities belong to the (physical) domain of definition ? How does the boundary condition $\Phi(\vartheta = \theta) = 0$ restrict the possible values for ν ? How do these values depend on θ ?

c) Consider a pointed tip with $180° - \theta = 5°$. Let the potential $\Phi(r = d, \vartheta = 0, \phi = 0) = \Phi_0$ at a small distance d. How large is the electric field closer to the tip at distance $d' = \frac{d}{10}$?