Theoretische Physik III: Klassische Elektrodynamik, Exercise 5

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1. Electric field of a dipole (Oral)

a) Recall the important result $\nabla^2 \frac{1}{|\mathbf{r}|} = -4\pi\delta^3(\mathbf{r})$ from Ex. 2.1 and generalize it to

$$\partial_\alpha \partial_\beta \frac{1}{|\mathbf{r}|} = -\frac{\delta_{\alpha\beta}}{|\mathbf{r}|^3} + 3\frac{x_\alpha x_\beta}{|\mathbf{r}|^5} - \frac{4\pi}{3}\delta_{\alpha\beta}\,\delta^3(\mathbf{r}). \tag{1}$$

Hint: Use a symmetry argument and the result from exercise 2.1 to derive the last term in equation (1).

b) In the lecture, it was demonstrated that the electric potential for a dipole $\mathbf{p}$ is given by $\phi(\mathbf{r}) = \frac{\mathbf{p}\cdot\mathbf{r}}{4\pi\epsilon_0|\mathbf{r}|^3} = -(\mathbf{p}\cdot\boldsymbol{\nabla})\frac{1}{4\pi\epsilon_0|\mathbf{r}|}$. Using relation (1), show that the electric field of the dipole can be written as ($\hat{\mathbf{r}} = \mathbf{r}/|\mathbf{r}|$):

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{3(\hat{\mathbf{r}}\cdot\mathbf{p})\,\hat{\mathbf{r}} - \mathbf{p}}{|\mathbf{r}|^3} - \frac{4\pi}{3}\mathbf{p}\,\delta^3(\mathbf{r})\right]. \tag{2}$$

The δ-function term in equation (2) is a correction for $\mathbf{r} = 0$. In the following, we are going to re-derive it in a different way to understand its physical origin.

Prove the following Theorem: The *average electric field* over a spherical volume of radius R, due to an arbitrary charge distribution within the sphere, is given by

$$\overline{\mathbf{E}} = -\frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}}{R^3}, \tag{3}$$

where $\mathbf{p}$ is the total dipole moment with respect to the center of the sphere.

c) To do this, first calculate the average electric field due to a single charge q at position $\mathbf{r}_q$ within the sphere (with volume V):

$$\overline{\mathbf{E}}_q = \frac{1}{V}\int_V \mathrm{d}^3r\,\mathbf{E}_q(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{V}\int_V \mathrm{d}^3r\,\frac{\mathbf{r}-\mathbf{r}_q}{|\mathbf{r}-\mathbf{r}_q|^3}. \tag{4}$$

Realize that this expression can also be considered as the electric field *at the position* $\mathbf{r}_q$, that is generated by a (fictional) sphere with a uniform charge density $\rho = q/V$. Use this analogy to calculate $\overline{\mathbf{E}}_q$ via Gauss's law.

d) Use the superposition principle to generalize the result for the point charge q to arbitrary charge distributions and prove equation (3).

e) Explicitly calculate the average electric field that is generated by a point-like dipole, by integrating the electric field from equation (2) over a sphere. In your integration, start by excluding a small region around the origin.

f) Finally, show that the δ-function term in equation (3) is essential to satisfy the average-value theorem.

Remark: Another approach is to calculate the electric field of a homogeneously polarized sphere of radius a. Outside of the sphere, the field is exactly given by equation (2). Inside the sphere, the field has a constant value $\mathbf{E}_{\text{in}} = -1/4\pi\epsilon_0 \cdot \mathbf{p}/a^3$, where $\mathbf{p}$ is the dipole moment of the sphere. As the size of the sphere goes to zero, the field strength goes to infinity in such a way that the integral over the sphere remains constant, giving the prefactor of the δ-function: $-\mathbf{p}/3\epsilon_0$.

2. Magnetic field of a finite coil (Written) [3pt]

Consider a wire coiled up cylindrically around the z-axis. Let R be the radius of this cylindrical coil and L its length (it starts at $z = -L/2$ and ends at $z = +L/2$). Let $n = N/L$ be the winding number per unit length and I be the (constant) current flowing through the wire. You may neglect boundary effects.

a) Calculate the z-component of the magnetic flux density B for points on the symmetry axis.

b) Determine the magnetic field for $L \to \infty$ at constant n.

Hint: $\int \mathrm{d}x \, \frac{1}{(x^2+w^2)^{3/2}} = \frac{x}{w^2\sqrt{x^2+w^2}}$

3. Spherical multipole moment (Oral)

The goal of this exercise is to calculate the spherical multipole moments q_{lm} of the following distribution of charges.

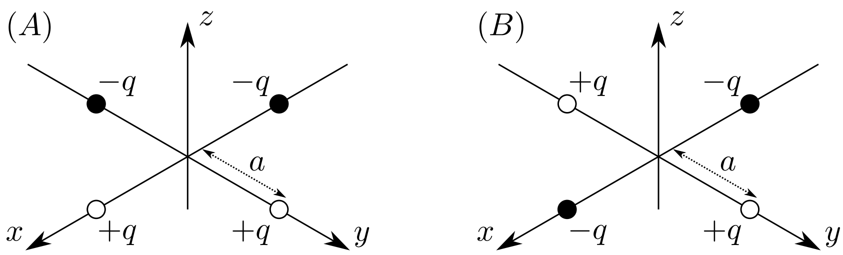

Figure 1: Two charge distributions (A) and (B) with four charges in the xy plane, placed at a distance a from the origin.

a) Write down the charge distribution $\rho(\mathbf{r})$ in spherical coordinates. The relation between the charge distribution in Cartesian coordinates and spherical coordinates is given by (why?):

$$\rho(r, \theta, \phi) = \frac{\rho(x, y, z)}{r^2 \sin\theta} \tag{5}$$

b) Compute the spherical monopole, dipole and quadrupole moments for both arrangements.