1. Electric field of a dipole (Oral)

a) Recall the important result \( \nabla^2 \frac{1}{|r|} = -4\pi \delta^3(r) \) from Ex. 2.1 and generalize it to

\[
\partial_\alpha \partial_\beta \frac{1}{|r|} = -\frac{\delta_{\alpha\beta}}{|r|^3} + \frac{3}{|r|^5} \delta_{\alpha\beta} \delta^3(r).
\]  

**Hint:** Use a symmetry argument and the result from exercise 2.1 to derive the last term in equation (1).

b) In the lecture, it was demonstrated that the electric potential for a dipole \( p \) is given by

\[
\phi(r) = \frac{p \cdot r}{4\pi \varepsilon_0 |r|^3} = -\left( p \cdot \nabla \right) \frac{1}{4\pi \varepsilon_0 |r|^3}. 
\]

Using relation (1), show that the electric field of the dipole can be written as (\( \hat{r} = r/|r| \)):

\[
E(r) = \frac{1}{4\pi \varepsilon_0} \left[ 3(\hat{r} \cdot p) \hat{r} - \frac{p}{|r|^3} \right].
\]

The \( \delta \)-function term in equation (2) is a correction for \( r = 0 \). In the following, we are going to re-derive it in a different way to understand its physical origin.

Prove the following **Theorem:** The average electric field over a spherical volume of radius \( R \), due to an arbitrary charge distribution within the sphere, is given by

\[
\overline{E} = -\frac{1}{4\pi \varepsilon_0} \frac{P}{R^3},
\]

where \( P \) is the total dipole moment with respect to the center of the sphere.

c) To do this, first calculate the average electric field due to a single charge \( q \) at position \( r_q \) within the sphere (with volume \( V \)):

\[
\overline{E}_q = \frac{1}{V} \int_V \mathrm{d}^3r \overline{E}_q(r) = \frac{1}{4\pi \varepsilon_0} \frac{q}{V} \int_V \mathrm{d}^3r \frac{r - r_q}{|r - r_q|^3}.
\]

Realize that this expression can also be considered as the electric field at the position \( r_q \), that is generated by a (fictional) sphere with a uniform charge density \( \rho = q/V \). Use this analogy to calculate \( \overline{E}_q \) via Gauss’s law.

d) Use the superposition principle to generalize the result for the point charge \( q \) to arbitrary charge distributions and prove equation (3).

e) Explicitly calculate the average electric field that is generated by a point-like dipole, by integrating the electric field from equation (2) over a sphere. In your integration, start by excluding a small region around the origin.
f) Finally, show that the \( \delta \)-function term in equation (3) is essential to satisfy the average-value theorem.

**Remark**: Another approach is to calculate the electric field of a homogeneously polarized sphere of radius \( a \). Outside of the sphere, the field is exactly given by equation (2). Inside the sphere, the field has a constant value \( E_{in} = -1/4\pi \epsilon_0 \cdot p/a^3 \), where \( p \) is the dipole moment of the sphere. As the size of the sphere goes to zero, the field strength goes to infinity in such a way that the integral over the sphere remains constant, giving the prefactor of the \( \delta \)-function: \(-p/3\epsilon_0\).

### 2. Magnetic field of a finite coil (Written) \([3pt]\)

Consider a wire coiled up cylindrically around the \( z \)-axis. Let \( R \) be the radius of this cylindrical coil and \( L \) its length (it starts at \( z = -L/2 \) and ends at \( z = +L/2 \)). Let \( n = N/L \) be the winding number per unit length and \( I \) be the (constant) current flowing through the wire. You may neglect boundary effects.

a) Calculate the \( z \)-component of the magnetic flux density \( B \) for points on the symmetry axis.

b) Determine the magnetic field for \( L \to \infty \) at constant \( n \).

**Hint**: \( \int dx \frac{1}{(x^2 + w^2)^{3/2}} = \frac{x}{w\sqrt{x^2 + w^2}} \)

### 3. Spherical multipole moment (Oral)

The goal of this exercise is to calculate the spherical multipole moments \( q_{lm} \) of the following distribution of charges.

![Figure 1: Two charge distributions (A) and (B) with four charges in the \( xy \) plane, placed at a distance \( a \) from the origin.](image)

a) Write down the charge distribution \( \rho(r) \) in spherical coordinates. The relation between the charge distribution in Cartesian coordinates and spherical coordinates is given by (why?):

\[
\rho(r, \theta, \phi) = \frac{\rho(x, y, z)}{r^2 \sin \theta}
\]  

b) Compute the spherical monopole, dipole and quadrupole moments for both arrangements.