# Theoretische Physik III: Klassische Elektrodynamik, Exercise 7

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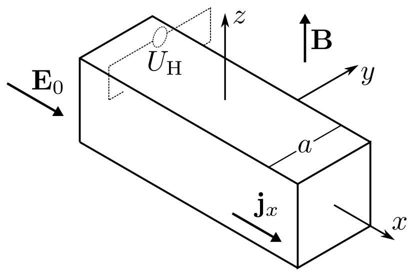
## 1. Hall effect (Oral)




Consider an infinitely long conductor in $x$-direction with finite extent $a$ in $y$- and $z$-direction. An electric field $\mathbf{E}_0 = E_x \mathbf{e}_x$ in $x$-direction is applied to the conductor, leading to a current density $\mathbf{j}_x$. An additional magnetic field $\mathbf{B} = B_z \mathbf{e}_z$ in $z$-direction will deflect charge carriers (electrons with charge $-e$) through the Lorentz force. This leads to an accumulation of charges on the sides of the material, resulting in an electric field $\mathbf{E}_{\mathrm{H}} = E_y \, \mathbf{e}_y$ along the $y$-direction. This phenomenon is called *Hall effect.*

a) Consider the stationary case ($j_y = v_y = 0$) and calculate the Hall field $\mathbf{E}_{\mathrm{H}}$ as well as the potential difference $U_{\mathrm{H}}$ between both sides in terms of the current density $j_x$. In which direction does $\mathbf{E}_{\mathrm{H}}$ point? *Hint:* The current density is given by $\mathbf{j} = -en\mathbf{v}$, where $n$ is the electron density and $\mathbf{v}$ their velocity. In the stationary case, there is no net force on the carriers in $y$-direction.

b) The Drude model for transport in metals assumes the following equation of motion for the charge carriers:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p} = -e\left(\mathbf{E} + \frac{1}{m}\mathbf{p} \times \mathbf{B}\right) - \frac{\mathbf{p}}{\tau}. \qquad (1)$$

Here, $\mathbf{p}$ is the momentum of the electrons, $m$ is the electron mass and $\tau$ is the relaxation time (scattering time). Motivate/derive this equation. *Hint:* The Drude model assumes that electrons undergo a scattering event with probability $\mathrm{d}t/\tau$ within an infinitesimal short time span $\mathrm{d}t$.

c) Determine the conductance tensor $\sigma_{kl}$, which is defined via: (Einstein convention)

$$j_k = \sigma_{kl} E_l, \qquad k, l \in \{x, y\} \qquad (2)$$

as well as the resistance tensor $\rho = \sigma^{-1}$ within the Drude model. *Hint:* Consider the stationary case ($\dot{\mathbf{p}} = 0$). Adopt the notation $\sigma_0 = ne^2\tau/m$ for the Drude DC conductance and $\omega_c = eB/m$ for the cyclotron frequency.

d) Calculate the Hall coefficient $R_{\mathrm{H}} = E_y/(j_x H_z)$ for the stationary case within the Drude model. What is the sign of $R_{\mathrm{H}}$? Is it possible to use the Hall effect to determine the charge of (unknown) particles/carriers of the electric current?

**2. Green's function for the wave equation (Oral)**

In the lecture, we have seen that the vector potential $\mathbf{A}(\mathbf{r},t)$ satisfies the following wave equation within Coulomb gauge (Coulomb-Eichung):

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\mathbf{A}(\mathbf{r},t) = -\mu_0\,\mathbf{j}_{\mathrm{t}}(\mathbf{r},t). \tag{3}$$

Here, $\mathbf{j}_{\mathrm{t}} = \mathbf{j}(\mathbf{r},t) - \epsilon_0\nabla\partial_t\phi(\mathbf{r},t)$ is the transversal part of the current density: We are going to solve equation (3) with the Green's function method. Let $G(\mathbf{r},t;\mathbf{r}',t')$ be the Green's function for the operator $\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$, i.e.

$$\Box\, G(\mathbf{r},t;\mathbf{r}',t') = 4\pi\delta^3(\mathbf{r}-\mathbf{r}')\delta(t-t'). \tag{4}$$

a) Write the general solution of Eq. (3) in terms of the Green's function.

b) Show that the Fourier transform of the Green's function is given by

$$G(\mathbf{k},\omega) = \frac{4\pi}{k^2 - \omega^2/c^2}. \tag{5}$$

c) As a function of momentum, $G(\mathbf{k},\omega)$ has poles at $k = |\mathbf{k}| = \pm\omega/c$. Calculate the Green's function $G^{\mathrm{r,a}}(\mathbf{r},t;\mathbf{r}',t')$ by inverse Fourier transformation. The indices 'r' and 'a' stand for the retarded and the advanced Green's function, respectively. The two functions are distinguished by the way the two poles are handled.

Transform from $\mathbf{k}$-space to real-space first. Use spherical coordinates and rewrite the integral over $k \in [0,\infty)$ into an integral over $k \in (-\infty,\infty)$. In which way can you close the integration contour in the complex $k$ plane? Shift the poles infinitesimally from the real axis by setting $\omega/c \longrightarrow \omega/c \pm i\epsilon$ (how does this affect the position of the poles?). Then, use the residue theorem to solve the $k$ integration. Finally, perform the transformation from the frequency domain back to the time domain.

The Green's function is called retarded (advanced) if the pole at $+\omega/c$ $(-\omega/c)$ is included in the contour.

**3. Gauge invariance of the classical equations of motion (Written) [3pt]**

The Lagrange function for a (non-relativistic) charged particle in an electromagnetic field is given by

$$\mathcal{L}(\mathbf{r},\dot{\mathbf{r}},t) = \frac{1}{2}m\dot{\mathbf{r}}^2 - q\,\phi(\mathbf{r},t) + q\,\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r},t). \tag{6}$$

Derive the equations of motion and show that they are gauge invariant (eichinvariant). Calculate the canonical momentum $\mathbf{p} = \partial\mathcal{L}/\partial\dot{\mathbf{r}}$. Is it gauge invariant? What is the relation between the mechanical and the canonical momentum? Show that the Hamiltonian is given by

$$\mathcal{H}(\mathbf{r},\mathbf{p},t) = \frac{1}{2m}\left(\mathbf{p} - q\mathbf{A}(\mathbf{r},t)\right)^2 + q\phi(\mathbf{r},t). \tag{7}$$