Theoretische Physik III: Klassische Elektrodynamik, Exercise 9

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1. Elliptically polarized waves (Oral)

A wave $\mathbf{E}(\mathbf{x}, t)$ with wave vector $\mathbf{k} = k\mathbf{e}_z$ is given by

$$\begin{aligned} E_x(\mathbf{x}, t) &= A\cos(kz - \omega t) \\ E_y(\mathbf{x}, t) &= B\cos(kz - \omega t + \phi). \end{aligned} \tag{1}$$

a) Show that the trajectory of the vector $\mathbf{E}(\mathbf{0}, t)$, which describes the polarization of the wave, is an ellipse. For which values of A, B and ϕ is this trajectory a circle ?
Hint: Use the trigonometric addition theorem

$$\cos(\omega t - \phi) = \cos(\omega t)\cos(\phi) + \sin(\omega t)\sin(\phi)$$

in order to transform the equations into the form of a conic section:

$$ax^2 + 2bxy + cy^2 + f = 0 \tag{2}$$

Which conditions does one need to impose on a, b, c ?

b) Show that for general A and B the wave can be written as the superposition of two oppositely circularly polarized waves

$$\mathbf{E}(\mathbf{x}, t) = \mathrm{Re}(\mathbf{E}_+(z, t) + \mathbf{E}_-(z, t)), \tag{3}$$

where $\mathbf{E}_\pm(z, t) = A_\pm \epsilon_\pm e^{i(kz - \omega t)}$. Here $A_\pm$ are constants and $\epsilon_\pm = \frac{1}{\sqrt{2}}(\mathbf{e}_x \pm i\mathbf{e}_y)$. Determine $A_\pm$ as a function of A, B and ϕ.
Hint: Write the wave as the real part of a complex vector and solve $\epsilon_\pm$ for $\mathbf{e}_x$ and $\mathbf{e}_y$.

2. Ideal wave guide (Written) [3pts]

We consider electromagnetic waves which are confined to an ideal cylindrical wave guide and propagate in the z-direction. For this geometry we can separate off the propagation in z-direction:

$$\begin{aligned} \mathbf{E}(x, y, z, t) &= \mathbf{E}_0(x, y)e^{i(kz - \omega t)} \\ \mathbf{B}(x, y, z, t) &= \mathbf{B}_0(x, y)e^{i(kz - \omega t)}. \end{aligned} \tag{4}$$

The boundary conditions at the interior boundary of the wave guide are $\mathbf{E}^{\|} = 0$ and $\mathbf{B}^{\perp} = 0$.

a) Derive equations for the x- and y-components of $\mathbf{E}$ and $\mathbf{B}$ from the Maxwell equations (depending on E_z and B_z) and show that

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \mu\epsilon\omega^2 - k^2\right] E_z = 0$$
$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \mu\epsilon\omega^2 - k^2\right] B_z = 0. \tag{5}$$

b) Show that in an ideal wave guide no transverse electromagnetic (TEM) modes exist.
Hint: If the z-component of the electric field $\mathbf{E}$ is equal to zero ($E_z = 0$), we have a transverse electric (TE) wave. For $B_z = 0$ it is called a transverse magnetic (TM) wave. If both $E_z = 0$ and $B_z = 0$, we have a transverse electromagnetic (TEM) wave.
Use Gauss' and Faraday's law as well as the boundary condition for $\mathbf{E}^{\parallel}$ to show that there are no TEM waves in a single, hollow, cylindrical conductor.

3. Coaxial cable (Oral)

A coaxial cable is a thin wire (radius a) surrounded by a conducting cylindrical shield (radius b > a). Write down solutions for TEM waves in the coaxial cable for a current I flowing through the thin wire.

Hint: The problem can be reduced to equations of magnetostatics and electrostatics in two dimensions.