1. Propagation of wave packets in non-linear media (Oral)

In this exercise we investigate the key concept of dispersion from a general point of view.

We start with a general, scalar field $\Psi$ the time evolution of which is given as a superposition of plain waves,

$$
\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{\Psi}_0(k) \exp(i[kx - \omega(k)t]) \, dk \tag{1}
$$

where $\hat{\Psi}_0(k) \equiv \mathcal{F}[\Psi_0](k)$ is the Fourier transform of the initial wave packet $\Psi_0 = \Psi(x,t=0)$. The function $\omega = \omega(k)$ is called dispersion relation and determined by the differential equation that governs the dynamics of $\Psi$.

a) Give two paradigmatic examples of differential equations (“wave equations”) with general solutions given by (1) and compare their corresponding dispersion relations $\omega = \omega(k)$.

*Hint: Quantum mechanics & Electrodynamics*

b) Assume that $\hat{\Psi}_0(k)$ is sharply peaked around $k_0$. Then it is reasonable to expand $\omega(k)$ at $k_0$ for small $k - k_0$ up to first order (Why?). Use this expansion with Eq. (1) to show that $\Psi(x,t)$ can be written in the form

$$
\Psi(x,t) = e^{i\phi(x-v_pt)} \cdot \psi(x-v_gt) \tag{2}
$$

where $\phi(x)$ is a real function and $\psi(x)$ an arbitrary scalar field. Give expressions for $v_p$ and $v_g$ in terms of $\omega(k)$. $v_p$ and $v_g$ are called phase- and group velocity, respectively.

In the following we focus on a special case, namely a Gaussian wave packet at $t = 0$:

$$
\Psi_0(x) \equiv \Psi(x,t=0) = \psi_0 \exp\left(-\frac{x^2}{2\sigma_x^2}\right) \tag{3}
$$

the propagation of which is still governed by an arbitrary dispersion relation $\omega = \omega(k)$. Here, $\sigma_x^2$ is the variance that describes the width of the wave packet.

c) Show that the initial wave packet in Fourier representation $\hat{\Psi}_0(k)$ is Gaussian as well, i.e.,

$$
\hat{\Psi}_0(k) = \hat{\psi}_0 \exp\left(-\frac{k^2}{2\sigma_k^2}\right) \tag{4}
$$
What is the relation of $\sigma_x$ and $\sigma_k$ and how can one interpret this result?

**Hint:**

$$\int_{\mathbb{R}} dx \ e^{-\frac{x^2}{2\sigma^2}} = \sqrt{2\pi\sigma} \quad \& \quad \text{completing the square} \tag{5}$$

d) Assume that $\hat{\Psi}_0(k)$ is peaked around $k_0 = ?$ so that an expansion of $\omega(k)$ up to second order in $k - k_0$ is a valid approximation (What is the requirement on $\sigma_x$ for $\hat{\Psi}_0(k)$ to be sharply “peaked”?):

$$\omega(k) \approx \omega_0 + v_g(k - k_0) + \frac{1}{2} w_g(k - k_0)^2 \tag{6}$$

$w_g$ is called *group velocity dispersion*. How does it relate to $v_g$?

Use Eq. (1) and your result from (c) to calculate $\Psi(x, t)$ explicitly.

e) Describe the qualitative behaviour of your solution $\Psi(x, t)$ for the following three dispersion relations:

$$\omega_1(k) = ck \quad \text{(EM wave in vacuum)}$$

$$\omega_2(k) = \frac{\hbar k^2}{2m} \quad \text{(Free quantum particle)}$$

$$\omega_3(k) = \sqrt{\omega_0^2 + c^2k^2} \quad \text{(EM wave in waveguides)} \tag{7a}$$
2. Energy transport in waveguides (Written) [4 pts]

In this exercise we focus on the energy flow in waveguides bounded by perfectly conducting walls and filled with a medium characterized by permittivity $\varepsilon$ and permeability $\mu$. The coordinate axes are oriented as shown in Fig. 1. After separating the propagating solution in $z$-direction, the fields $E$ and $B$ satisfy the eigenvalue equation

$$\left[\nabla_t^2 + \gamma^2_{\lambda}\right] \begin{pmatrix} E(x, y) \\ B(x, y) \end{pmatrix} = 0$$

(8)

with the additional constraints

$$E_z|_{\partial\mathcal{V}} = 0, \quad H_z = 0$$  \hspace{0.5cm} (TM modes)  \hspace{0.5cm} (9a)$$
$$\partial_n H_z|_{\partial\mathcal{V}} = 0, \quad E_z = 0$$  \hspace{0.5cm} (TE modes)  \hspace{0.5cm} (9b)

where $\gamma^2_{\lambda} = \mu\varepsilon\omega^2/c^2 - k^2_{\lambda}$, $\nabla_t \equiv e_x\partial_x + e_y\partial_y$ and $\partial\mathcal{V}$ denotes the boundary (“walls”) of the waveguide. Introducing the critical frequency $\omega_{\lambda} \equiv \frac{c}{\sqrt{\mu\varepsilon}\gamma_{\lambda}}$ allows us to write $k^2_{\lambda} = \frac{\mu\varepsilon}{c^2}(\omega^2 - \omega_{\lambda}^2)$ for the wave number that describes the propagation along the $z$-axis of the waveguide.

In the lecture it was shown that the solutions for the transversal field components $\Psi_t = \Psi_x e_x + \Psi_y e_y$ are determined by the solutions for the $z$-components $\Psi_z$ via

$$E_t = \frac{ik_{\lambda}}{\gamma^2_{\lambda}} \nabla_t E_z, \quad H_t = \frac{\varepsilon\omega}{ck_{\lambda}} e_z \times E_t$$  \hspace{0.5cm} (TM modes),  \hspace{0.5cm} (10a)$$
$$H_t = \frac{ik_{\lambda}}{\gamma^2_{\lambda}} \nabla_t H_z, \quad E_t = -\frac{\mu\omega}{ck_{\lambda}} e_z \times H_t$$  \hspace{0.5cm} (TE modes).  \hspace{0.5cm} (10b)

The flow of energy is given by the (complex) pointing vector

$$S = \frac{c}{8\pi} E \times H^*$$

(11)

where $*$ denotes complex conjugation.
a) Employ the solutions given above to show that the pointing vector takes the form

\[
S = \frac{\omega k_{\lambda}}{8\pi \gamma_{\lambda}} \begin{cases} 
\varepsilon [\nabla t E_z]^2 e_z + i \frac{\gamma^2}{k_{\lambda}^2} E_z \nabla t E^*_z] & \text{(TM modes)} \\
\mu [\nabla t H_z]^2 e_z - i \frac{\gamma^2}{k_{\lambda}^2} H_z \nabla t H^*_z] & \text{(TE modes)}
\end{cases}
\]

(12)

b) Which contribution in Eq. (12) determines the energy flow in z-direction? Integrate this part over the cross section \(S\) of the waveguide for both TE and TM modes and show that the propagating power is given by

\[
\begin{bmatrix} P_{TM} \\ P_{TE} \end{bmatrix} = \frac{c}{8\pi \sqrt{\mu \varepsilon}} \left( \frac{\omega}{\omega_{\lambda}} \right)^2 \sqrt{1 - \frac{\omega^2}{\omega_{\lambda}^2}} \left\{ \varepsilon \int_S |E_z|^2 \right\} \left\{ \mu \int_S |H_z|^2 \right\} \cdot (13)
\]

Hint: Use Green’s first identity for two scalar fields \(\Psi\) and \(\Phi\)

\[
\int_U dV \left[ \Phi \nabla^2 \Psi + \nabla \Phi \cdot \nabla \Psi \right] = \oint_{\partial U} dA \Phi \frac{\partial \Psi}{\partial n} \quad (14)
\]

and the boundary conditions given in (9). Here, \(U \subset \mathbb{R}^n\) is some \(n\)-dimensional subset, \(\partial U\) its boundary, and \(\partial_n \Psi = n \cdot \nabla \Psi\) is the normal derivative with respect to \(\partial U\). Eq. (8) may be useful as well.

c) Along the same lines, calculate the energy \(U_{TM/TE}\) per unit length of the waveguide and show that

\[
\begin{bmatrix} U_{TM} \\ U_{TE} \end{bmatrix} = \frac{1}{8\pi} \left( \frac{\omega}{\omega_{\lambda}} \right)^2 \left\{ \varepsilon \right\} \int_S dA \left\{ |E_z|^2 \right\} \left\{ \mu \int_S |H_z|^2 \right\} \cdot (15)
\]

Hint: The time-averaged energy \(u\) per volume (energy density) is given by

\[
u = \frac{1}{16\pi} \left( \varepsilon |E|^2 + \mu |H|^2 \right) \quad (16)
\]

d) Finally, combine the results (13) and (15) to derive an expression for the velocity of the energy flux and compare your result with the group velocity \(v_g = \frac{d\omega}{dk_{\lambda}}\).