Theoretische Physik III: Klassische Elektrodynamik, Exercise 10

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1. Propagation of wave packets in non-linear media (Oral)

In this exercise we investigate the key concept of *dispersion* from a general point of view.

We start with a general, scalar field Ψ the time evolution of which is given as a superposition of plain waves,

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{d}k\, \hat{\Psi}_0(k) \exp\left(i[kx - \omega(k)t]\right) \tag{1}$$

where $\hat{\Psi}_0(k) \equiv \mathcal{F}[\Psi_0](k)$ is the Fourier transform of the initial wave packet $\Psi_0 = \Psi(x, t = 0)$. The function $\omega = \omega(k)$ is called *dispersion relation* and determined by the differential equation that governs the dynamics of Ψ.

a) Give two paradigmatic examples of differential equations ("wave equations") with general solutions given by (1) and compare their corresponding dispersion relations $\omega = \omega(k)$.

Hint: Quantum mechanics & Electrodynamics

b) Assume that $\hat{\Psi}_0(k)$ is sharply peaked around k_0. Then it is reasonable to expand $\omega(k)$ at k_0 for small $k - k_0$ up to first order (Why?). Use this expansion with Eq. (1) to show that $\Psi(x,t)$ can be written in the form

$$\Psi(x,t) = e^{i\phi(x - v_p t)} \cdot \psi(x - v_g t) \tag{2}$$

where $\phi(x)$ is a real function and $\psi(x)$ an arbitrary scalar field. Give expressions for v_p and v_g in terms of $\omega(k)$. v_p and v_g are called *phase*- and *group* velocity, respectively.

In the following we focus on a special case, namely a Gaussian wave packet at $t = 0$:

$$\Psi_0(x) \equiv \Psi(x, t = 0) = \psi_0 \exp\left(-\frac{x^2}{2\sigma_x^2}\right), \tag{3}$$

the propagation of which is still governed by an arbitrary dispersion relation $\omega = \omega(k)$. Here, σ_x^2 is the variance that describes the width of the wave packet.

c) Show that the initial wave packet in Fourier representation $\hat{\Psi}_0(k)$ is Gaussian as well, i.e.,

$$\hat{\Psi}_0(k) = \hat{\psi}_0 \exp\left(-\frac{k^2}{2\sigma_k^2}\right). \tag{4}$$

What is the relation of σ_x and σ_k and how can one interpret this result?

Hint:

$$\int_{\mathbb{R}} \mathrm{d}x\, e^{-\frac{x^2}{2\sigma^2}} = \sqrt{2\pi}\sigma \quad \& \quad \text{completing the square} \tag{5}$$

d) Assume that $\hat{\Psi}_0(k)$ is peaked around $k_0 =?$ so that an expansion of $\omega(k)$ up to second order in $k - k_0$ is a valid approximation (What is the requirement on σ_x for $\hat{\Psi}_0(k)$ to be sharply "peaked"?):

$$\omega(k) \approx \omega_0 + v_g(k - k_0) + \frac{1}{2}\, w_g(k - k_0)^2 \tag{6}$$

w_g is called *group velocity dispersion.* How does it relate to v_g?
Use Eq. (1) and your result from (c) to calculate $\Psi(x,t)$ explicitely.

e) Describe the qualitative behaviour of your solution $\Psi(x,t)$ for the following three dispersion relations:

$$\begin{aligned} \omega_1(k) &= ck && \text{(EM wave in vacuum)} \\ \omega_2(k) &= \frac{\hbar k^2}{2m} && \text{(Free quantum particle)} \\ \omega_3(k) &= \sqrt{\omega_0^2 + c^2k^2} && \text{(EM wave in waveguides)} \end{aligned} \tag{7a}$$

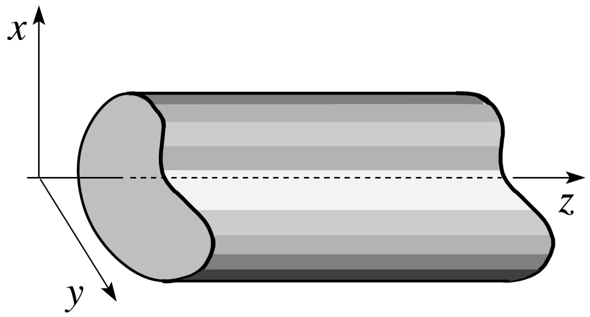

Figure 1: The waveguide is translationally invariant in z-direction.

2. Energy transport in waveguides (Written) [4 pts]

In this exercise we focus on the energy flow in waveguides bounded by perfectly conducting walls and filled with a medium characterized by permittivity ε and permeability μ. The coordinate axes are oriented as shown in Fig. 1. After separating the propagating solution in z-direction, the fields $\mathbf{E}$ and $\mathbf{B}$ satisfy the eigenvalue equation

$$\left[\nabla_t^2 + \gamma_\lambda^2\right] \begin{Bmatrix} \mathbf{E}(x,y) \\ \mathbf{B}(x,y) \end{Bmatrix} = 0 \tag{8}$$

with the additional constraints

$$E_z|_{\partial\mathcal{V}} = 0\,, \quad H_z = 0 \qquad \text{(TM modes)} \tag{9a}$$

$$\partial_n H_z|_{\partial\mathcal{V}} = 0\,, \quad E_z = 0 \qquad \text{(TE modes)} \tag{9b}$$

where $\gamma_\lambda^2 = \mu\varepsilon\omega^2/c^2 - k_\lambda^2$, $\nabla_t \equiv \mathbf{e}_x\partial_x + \mathbf{e}_y\partial_y$ and $\partial\mathcal{V}$ denotes the boundary (“walls”) of the waveguide. Introducing the critical frequency $\omega_\lambda \equiv \frac{c}{\sqrt{\mu\varepsilon}}\gamma_\lambda$ allows us to write $k_\lambda^2 = \frac{\mu\varepsilon}{c^2}(\omega^2 - \omega_\lambda^2)$ for the wave number that describes the propagation along the z-axis of the waveguide.

In the lecture it was shown that the solutions for the transversal field components $\mathbf{\Psi}_t = \Psi_x\mathbf{e}_x + \Psi_y\mathbf{e}_y$ are determined by the solutions for the z-components Ψ_z via

$$\mathbf{E}_t = \frac{ik_\lambda}{\gamma_\lambda^2}\nabla_t E_z, \quad \mathbf{H}_t = \frac{\varepsilon\omega}{ck_\lambda}\mathbf{e}_z \times \mathbf{E}_t \qquad \text{(TM modes)}\,, \tag{10a}$$

$$\mathbf{H}_t = \frac{ik_\lambda}{\gamma_\lambda^2}\nabla_t H_z, \quad \mathbf{E}_t = -\frac{\mu\omega}{ck_\lambda}\mathbf{e}_z \times \mathbf{H}_t \qquad \text{(TE modes)}\,. \tag{10b}$$

The flow of energy is given by the (complex) pointing vector

$$\mathbf{S} = \frac{c}{8\pi}\mathbf{E} \times \mathbf{H}^* \tag{11}$$

where $*$ denotes complex conjugation.

a) Employ the solutions given above to show that the pointing vector takes the form

$$\mathbf{S} = \frac{\omega k_\lambda}{8\pi\gamma_\lambda^4} \begin{cases} \varepsilon[|\nabla_t E_z|^2 \mathbf{e}_z + i\frac{\gamma_\lambda^2}{k_\lambda} E_z \nabla_t E_z^*] \,, & \text{(TM modes)} \\ \mu[|\nabla_t H_z|^2 \mathbf{e}_z - i\frac{\gamma_\lambda^2}{k_\lambda} H_z^* \nabla_t H_z] \,. & \text{(TE modes)} \end{cases} \tag{12}$$

b) Which contribution in Eq. (12) determines the energy flow in z-direction? Integrate this part over the cross section S of the waveguide for both TE and TM modes and show that the propagating power is given by

$$\begin{Bmatrix} P_{\text{TM}} \\ P_{\text{TE}} \end{Bmatrix} = \frac{c}{8\pi\sqrt{\mu\varepsilon}} \left(\frac{\omega}{\omega_\lambda}\right)^2 \sqrt{1 - \frac{\omega_\lambda^2}{\omega^2}} \begin{Bmatrix} \varepsilon \\ \mu \end{Bmatrix} \int_S \mathrm{d}A \begin{Bmatrix} |E_z|^2 \\ |H_z|^2 \end{Bmatrix} . \tag{13}$$

Hint: Use Green's first identity for two scalar fields Ψ and Φ

$$\int_U \mathrm{d}V \left[\Phi\nabla^2\Psi + \nabla\Phi \cdot \nabla\Psi\right] = \oint_{\partial U} \mathrm{d}A \, \Phi \frac{\partial\Psi}{\partial n} \tag{14}$$

and the boundary conditions given in (9). Here, $U \subset \mathbb{R}^n$ is some n-dimensional subset, ∂U its boundary, and $\partial_n \Psi \equiv \mathbf{n} \cdot \nabla\Psi$ is the normal derivative with respect to ∂U. Eq. (8) may be useful as well.

c) Along the same lines, calculate the energy $U_{\text{TM/TE}}$ *per unit length* of the waveguide and show that

$$\begin{Bmatrix} U_{\text{TM}} \\ U_{\text{TE}} \end{Bmatrix} = \frac{1}{8\pi} \left(\frac{\omega}{\omega_\lambda}\right)^2 \begin{Bmatrix} \varepsilon \\ \mu \end{Bmatrix} \int_S \mathrm{d}A \begin{Bmatrix} |E_z|^2 \\ |H_z|^2 \end{Bmatrix} . \tag{15}$$

Hint: The time-averaged energy u *per volume* (energy density) is given by

$$u = \frac{1}{16\pi} \left(\varepsilon|\mathbf{E}|^2 + \mu|\mathbf{H}|^2\right) . \tag{16}$$

d) Finally, combine the results (13) and (15) to derive an expression for the velocity of the energy flux and compare your result with the group velocity $v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k_\lambda}$.