Theoretische Physik III: Klassische Elektrodynamik, Exercise 11

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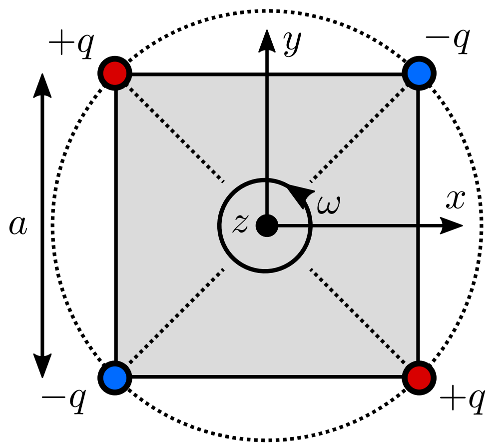
1. Rotating quadrupole (Written) [3 pts]

Consider the quadrupole setup depicted below, with two pairs of opposing charges $\pm q$ fixed at the corners of a square of size a. The square lies in the xy-plane and rotates with $\boldsymbol{\omega} = \omega \mathbf{e}_z$ around its center:



In the following we focus on the electromagnetic waves radiated by this setup.

a) Write down the time-dependent charge distribution $\rho(\mathbf{x}, t)$ and calculate the quadrupole tensor

$$Q_{ij}(t) = \int_{\mathbb{R}^3} \mathrm{d}^3x \, \rho(\mathbf{x}, t) \left(3x_i x_j - |\mathbf{x}|^2 \delta_{ij}\right) . \tag{1}$$

To this end, consider the initial condition where the $+q$ charges are located on the x-axis for $t = 0$.

Result:

$$\begin{aligned} Q_{3i} &= Q_{i3} = 0, \;\; i = 1, 2, 3 \\ Q_{11} &= -Q_{22} = 3qa^2 \; \mathrm{Re}\, e^{-2i\omega t} \\ Q_{21} &= Q_{12} = 3qa^2 \; \mathrm{Re}\, ie^{-2i\omega t} \end{aligned}$$

b) Show that the general expression for the angular power distribution in the far-field reads

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{c}{288\pi} k^6 \left|\hat{\mathbf{r}} \times Q_0 \hat{\mathbf{r}}\right|^2 \tag{2}$$

with $\hat{\mathbf{r}} = \mathbf{r}/|\mathbf{r}|$. Here, Q_0 is the 3×3 amplitude matrix (quadrupole tensor) defined by components Q_{ij} (without the oscillating factor).

Hint: The fields in the far-field approximation are given by

$$\mathbf{B} = -\frac{ik^3}{6}\frac{e^{ikr}}{r}\,\hat{\mathbf{r}} \times Q_0\hat{\mathbf{r}} \quad \text{and} \quad \mathbf{E} = \mathbf{B} \times \hat{\mathbf{r}}\,, \tag{3}$$

and the angular power distribution reads

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{cr^2}{8\pi}\,\hat{\mathbf{r}} \cdot (\mathbf{E} \times \mathbf{B}^*)\ . \tag{4}$$

c) Evaluate Eq. (2) with $k = 2\omega/c$ (why 2ω?) in spherical coordinates for the setup at hand using your result in a). Compare the result with the angular power distribution of an oscillating *dipole*.

2. Spherical Bessel functions (Oral)

In the course of your discussion of antennas in the lecture, the *spherical* Bessel and Hankel functions j_l and $h_l^{(1)} \equiv h_l^+$ played a crucial role for the expansion of the vector potential. In this exercise you will derive explicit expressions for these functions.

The spherical Bessel equation describes the radial part $R_l(r)$ of the solution $\Phi(r,\theta,\varphi) = R_l(r)Y_{lm}(\theta,\varphi)$ of the Helmoltz equation $[\Delta + k^2]\Phi = 0$ and reads

$$\left[\partial_x^2 + \frac{2}{x}\partial_x + \left(1 - \frac{l(l+1)}{x^2}\right)\right] R_l(x) = 0 \qquad \text{for} \qquad l \in \mathbb{N}_0 \tag{5}$$

with $x = kr$.

a) As a warm-up, show that the substitution $R_l(x) = \frac{u_l(x)}{\sqrt{x}}$ converts the spherical Bessel equation to the ordinary Bessel equation

$$\left[\partial_x^2 + \frac{1}{x}\partial_x + \left(1 - \frac{\nu^2}{x^2}\right)\right] u_l(x) = 0 \tag{6}$$

with half integer $\nu = l + \frac{1}{2}$. Give solutions of (5) in terms of the Bessel and Neumann functions $J_\nu(x)$ and $N_\nu(x)$ which have been introduced in the lecture some time ago during the discussion of electrostatics.

The solutions derived from $J_\nu(x)$ and $N_\nu(x)$ are denoted as $j_l(x)$ and $n_l(x)$ and referred to as *spherical* Bessel and Neumann functions, respectively (up to normalizing factors). In the remainder of this exercise, you derive explicit expressions for these functions.

b) To this end, prove that the *spherical Hankel functions*

$$h_l^\pm(x) = \mp\frac{(x/2)^l}{l!}\int_{\pm 1}^{i\infty} \mathrm{d}t\, e^{ixt}(1-t^2)^l \tag{7}$$

are solutions of (5) for $x > 0$ and $l \in \mathbb{N}_0$.

Hint: Use $x^{-1}\partial_x^2 x = 2x^{-1}\partial_x + \partial_x^2$ and write the integrand as a total derivative with respect to t.

Please note reverse side!

c) Now show that $h_l^\pm$ satisfy the recursion relation

$$\frac{\mathrm{d}h_l^\pm(x)}{\mathrm{d}x} = \frac{l}{x}h_l^\pm(x) - h_{l+1}^\pm(x)\,. \tag{8}$$

The spherical Hankel functions are a basis of the two-dimensional solution space for every l. Another common basis is given by the linear combinations

$$j_l(x) = \frac{1}{2}[h_l^+(x) + h_l^-(x)] \qquad \text{and} \qquad n_l(x) = \frac{1}{2i}[h_l^+(x) - h_l^-(x)] \tag{9}$$

which are the *spherical* Bessel and Neumann functions introduced in a).

d) Use the recursion in c) to prove the explicit expressions

$$j_l(x) \;=\; (-x)^l\left(\frac{1}{x}\frac{d}{dx}\right)^l\frac{\sin(x)}{x} \tag{10a}$$

$$n_l(x) \;=\; -(-x)^l\left(\frac{1}{x}\frac{d}{dx}\right)^l\frac{\cos(x)}{x} \tag{10b}$$

These are known as *Rayleigh's formulas.*

Hint: Mathematical induction.

e) Use the above results to write down $j_l(x), n_l(x)$ and $h_l^+(x), h_l^-(x)$ for $l = 0, 1$ and sketch the graphs of $j_l(x), n_l(x)$.

3. Lifetime of "classical" atoms (Oral)

In the lecture it was shown that an oscillating dipole $\mathbf{p}(t) = \mathbf{p}_0 e^{i\omega t}$ radiates the power

$$P_{\mathrm{Dipole}} = \frac{1}{4\pi\varepsilon_0}\frac{ck^4}{3}|\mathbf{p}_0|^2 \tag{11}$$

with vacuum dispersion $ck = \omega$.

Pretend you forgot everything you have learned in your quantum mechanics course last semester and think of the hydrogen atom as a positive central charge $+e$ with mass m_p (the proton) orbited classically by a charge with opposite sign $-e$ and mass m_e (the electron). Due to $m_p \gg m_e$ you may consider the proton stationary, $\mathbf{r}_p(t) \equiv \mathbf{0}$, and the electron's position parametrized by $\mathbf{r}_e(t) = a_B\cos(\omega_e t)\mathbf{e}_x + a_B\sin(\omega_e t)\mathbf{e}_y$ where $a_B = \frac{\hbar^2}{m_e e^2}$ is the Bohr radius.

Starting from (11), derive an expression for the radiated power of this "classical" atom in terms of ω_e and a_B. Get an estimate for ω_e using Coulomb's law and plug in the numbers to derive the timescale that governs the lifetime τ of this system. Are these results consistent with your experience? How does quantum mechanics mend matters?