1. Rotating quadrupole (Written) [3 pts]

Consider the quadrupole setup depicted below, with two pairs of opposing charges $\pm q$ fixed at the corners of a square of size $a$. The square lies in the $xy$-plane and rotates with $\omega = \omega e_z$ around its center:

In the following we focus on the electromagnetic waves radiated by this setup.

a) Write down the time-dependent charge distribution $\rho(x, t)$ and calculate the quadrupole tensor

$$Q_{ij}(t) = \int_{\mathbb{R}^3} d^3x \rho(x, t) \left( 3x_i x_j - |x|^2 \delta_{ij} \right). \quad (1)$$

To this end, consider the initial condition where the $+q$ charges are located on the $x$-axis for $t = 0$.

Result:

$$Q_{3i} = Q_{i3} = 0, \ i = 1, 2, 3$$
$$Q_{11} = -Q_{22} = 3qa^2 \ Re e^{-2i\omega t}$$
$$Q_{21} = Q_{12} = 3qa^2 \ Re ie^{-2i\omega t}$$

b) Show that the general expression for the angular power distribution in the far-field reads

$$\frac{dP}{d\Omega} = \frac{c}{288\pi} k^6 |\hat{r} \times Q_0 \hat{r}|^2 \quad (2)$$
with \( \hat{r} = r/|r| \). Here, \( Q_0 \) is the \( 3 \times 3 \) amplitude matrix (quadrupole tensor) defined by components \( Q_{ij} \) (without the oscillating factor).

*Hint:* The fields in the far-field approximation are given by
\[
B = -\frac{ik^3 e^{ikr}}{6r} \hat{r} \times Q_0 \hat{r} \quad \text{and} \quad E = B \times \hat{r},
\] and the angular power distribution reads
\[
\frac{dP}{d\Omega} = \frac{cr^2}{8\pi} \hat{r} \cdot (E \times B^*) .
\] (4)

c) Evaluate Eq. (2) with \( k = 2\omega/c \) (why \( 2\omega \)?) in spherical coordinates for the setup at hand using your result in a). Compare the result with the angular power distribution of an oscillating dipole.

2. Spherical Bessel functions (Oral)

In the course of your discussion of antennas in the lecture, the spherical Bessel and Hankel functions \( j_l \) and \( h_l^{(1)} \equiv h_l^+ \) played a crucial role for the expansion of the vector potential. In this exercise you will derive explicit expressions for these functions.

The spherical Bessel equation describes the radial part \( R_l(x) \) of the solution \( \Phi(r, \theta, \varphi) = R_l(r)Y_{lm}(\theta, \varphi) \) of the Helmholtz equation \( \left[ \Delta + k^2 \right] \Phi = 0 \) and reads
\[
\left[ \partial_x^2 + \frac{2}{x} \partial_x + \left( 1 - \frac{(l + 1)}{x^2} \right) \right] R_l(x) = 0 \quad \text{for} \quad l \in \mathbb{N}_0
\] (5)

with \( x = kr \).

a) As a warm-up, show that the substitution \( R_l(x) = \frac{u_l(x)}{\sqrt{x}} \) converts the spherical Bessel equation to the ordinary Bessel equation
\[
\left[ \partial_x^2 + \frac{1}{x} \partial_x + \left( 1 - \frac{\nu^2}{x^2} \right) \right] u_l(x) = 0
\] (6)

with half integer \( \nu = l + \frac{1}{2} \). Give solutions of (5) in terms of the Bessel and Neumann functions \( J_\nu(x) \) and \( N_\nu(x) \) which have been introduced in the lecture some time ago during the discussion of electrostatics.

The solutions derived from \( J_\nu(x) \) and \( N_\nu(x) \) are denoted as \( j_l(x) \) and \( n_l(x) \) and referred to as spherical Bessel and Neumann functions, respectively (up to normalizing factors). In the remainder of this exercise, you derive explicit expressions for these functions.

b) To this end, prove that the spherical Hankel functions
\[
h_l^\pm(x) = \mp \frac{(x/2)^l}{l!} \int_{\pm 1}^{\infty} dt e^{\pm x t}(1 - t^2)^l
\] (7)

are solutions of (5) for \( x > 0 \) and \( l \in \mathbb{N}_0 \).

*Hint:* Use \( x^{-1} \partial_x^2 x = 2x^{-1} \partial_x + \partial_x^2 \) and write the integrand as a total derivative with respect to \( t \).

Please note reverse side!
c) Now show that \( h_l^\pm \) satisfy the recursion relation
\[
\frac{dh_l^\pm(x)}{dx} = \frac{l}{x} h_l^\pm(x) - h_{l+1}^\pm(x).
\] (8)

The spherical Hankel functions are a basis of the two-dimensional solution space for every \( l \). Another common basis is given by the linear combinations
\[
j_l(x) = \frac{1}{2} [h_l^+(x) + h_l^-(x)] \quad \text{and} \quad n_l(x) = \frac{1}{2i} [h_l^+(x) - h_l^-(x)]
\] (9)
which are the spherical Bessel and Neumann functions introduced in a).

d) Use the recursion in c) to prove the explicit expressions
\[
j_l(x) = (-x)^l \left( \frac{1}{x} \frac{d}{dx} \right)^l \frac{\sin(x)}{x} \quad (10a)
n_l(x) = -(-x)^l \left( \frac{1}{x} \frac{d}{dx} \right)^l \frac{\cos(x)}{x} \quad (10b)
\]
These are known as Rayleigh’s formulas.

Hint: Mathematical induction.

e) Use the above results to write down \( j_l(x), n_l(x) \) and \( h_l^+(x), h_l^-(x) \) for \( l = 0, 1 \) and sketch the graphs of \( j_l(x), n_l(x) \).

3. Lifetime of “classical” atoms (Oral)

In the lecture it was shown that an oscillating dipole \( \mathbf{p}(t) = p_0 e^{i\omega t} \) radiates the power
\[
P_{\text{Dipole}} = \frac{1}{4\pi\varepsilon_0} \frac{c k^4}{3} |p_0|^2
\] (11)
with vacuum dispersion \( ck = \omega \).

Pretend you forgot everything you have learned in your quantum mechanics course last semester and think of the hydrogen atom as a positive central charge \(+e\) with mass \( m_p \) (the proton) orbited classically by a charge with opposite sign \(-e\) and mass \( m_e \) (the electron). Due to \( m_p \gg m_e \) you may consider the proton stationary, \( \mathbf{r}_p(t) \equiv 0 \), and the electron’s position parametrized by \( \mathbf{r}_e(t) = a_B \cos(\omega_e t) \mathbf{e}_x + a_B \sin(\omega_e t) \mathbf{e}_y \) where \( a_B = \frac{\hbar^2}{m_e e^2} \) is the Bohr radius.

Starting from (11), derive an expression for the radiated power of this “classical” atom in terms of \( \omega_e \) and \( a_B \). Get an estimate for \( \omega_e \) using Coulomb’s law and plug in the numbers to derive the timescale that governs the lifetime \( \tau \) of this system. Are these results consistent with your experience? How does quantum mechanics mend matters?