Theoretische Physik III: Klassische Elektrodynamik, Exercise 13

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1. Lorentz Group (Oral)

In this exercise, we prove that the Lorentz group is a group as well as we analyze its properties. First, let us revise the definition of a group.

Definition: A group is a set G together with an operation $\bullet$ (called the *group law* of G) that combines any two elements a and b to form another element, denoted by $a \bullet b$. To qualify as a group, the set and the operation, $(G, \bullet)$, must satisfy four requirements known as the *group axioms*:

1) **Closure**: For all $a, b \in G$, the result of the operation $a \bullet b$ is also an element of G.
2) **Associativity**: For all $a, b, c \in G$ the following relation is satisfied $(a \bullet b) \bullet c = a \bullet (b \bullet c)$.
3) **Identity element**: There exists an element $e \in G$, such that for every element $a \in G$, the equality $e \bullet a = a \bullet e = a$ holds. Such an element is unique, and thus called *the* identity element.
4) **Inverse element**: For each $a \in G$, there exists an element $b \in G$ such that $a \bullet b = b \bullet a = e$, where e is the identity element.

In the lecture, for Lorentz transformations, we defined G as the set characterized by invariance of the metric tensor g of Minkowski spacetime, i.e.

$$G := \{\Lambda \in \mathbb{R}^{4\times 4} \quad | \quad \Lambda^t g \Lambda = g\} \tag{1}$$

and the group operation $\bullet$ as the multiplication of matrices.

a) Show that the Lorentz group is a group.

b) Depending on $\det(\Lambda)$ and $\operatorname{sign}(\Lambda^0{}_0)$ the Lorentz group can be divided into four components. Show that *proper orthochronous Lorentz transformations*, i.e. $\det(\Lambda) = 1$ and $\operatorname{sign}(\Lambda^0{}_0) = 1$, form a group.

c) Next, proof that each of three other components does not form a group. Give an example of the combination of two out of four components which again gives a proper group.

2. Galilean invariance of handicapped Maxwell equations (Oral)

In the lecture we learned that Maxwell's equations are not invariant under Galilean transformations $\mathbf{r}' = \mathbf{r} + \mathbf{v}t$, $t' = t$. Here, we will show that Maxwell's equations become Galilean invariant if one drops the induction term in the Maxwell-Faraday equation

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= \rho/\epsilon_0, && (2)\\
\nabla \times \mathbf{E} &= -\underbrace{\cancel{\partial_t \mathbf{B}}}_{0}, && (3)\\
\nabla \cdot \mathbf{B} &= 0, && (4)\\
\nabla \times \mathbf{B} &= \mu_0(\mathbf{j} + \epsilon_0 \partial \mathbf{E}/\partial t). && (5)
\end{aligned}$$

For this purpose show that above equations are form invariant if we define $\mathbf{E}', \mathbf{B}'$ by

$$\mathbf{E}' = \mathbf{E}, \qquad \mathbf{B}' = \mathbf{B} + \epsilon_0\mu_0 \mathbf{v} \times \mathbf{E}. \tag{6}$$

Hint: You might find the following relations useful:

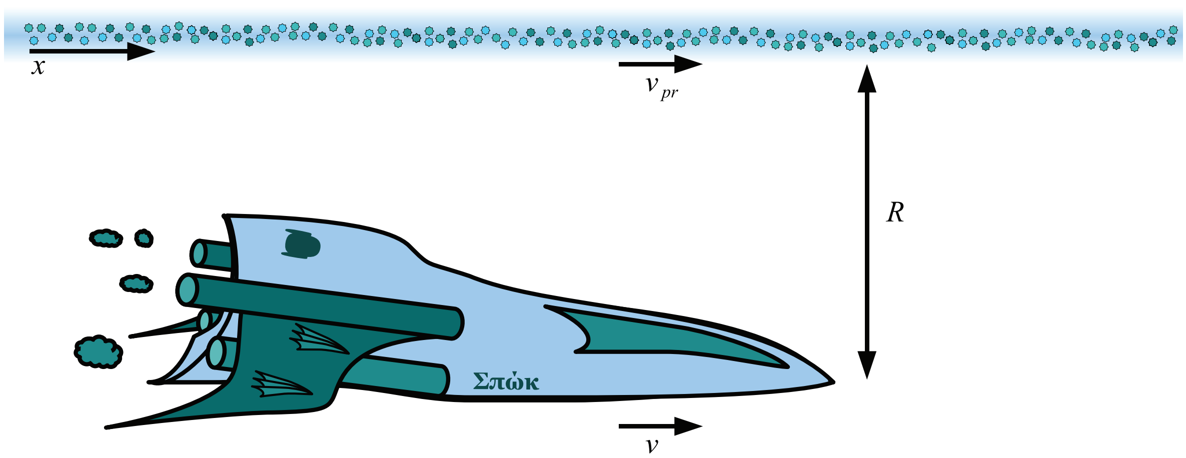
$$\begin{aligned}
\nabla \cdot (\mathbf{a} \times \mathbf{b}) &= \mathbf{b} \cdot \nabla \times \mathbf{a} - \mathbf{a} \cdot \nabla \times \mathbf{b}, && (7)\\
\nabla \times (\mathbf{a} \times \mathbf{b}) &= \mathbf{a}\nabla \cdot \mathbf{b} - \mathbf{b}\nabla \cdot \mathbf{a} + (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b}. && (8)
\end{aligned}$$

3. Tractor beam (Oral)

A beam of protons flies along the x-axis, wherein the protons have a speed $v_{pr} = c/3$ and a density n (in protons per unit length).

a) Use the laws of electrostatics and magnetostatics to calculate the fields $\mathbf{E}(r)$ and $\mathbf{B}(r)$ generated by the beam as a function of the distance to the beam r.
Tip: The fields produced by the beam are given by $I = env_{pr}$ and the charge distribution can be approximated for $r \gg 1/n$ as continuous, what leads to the charge density $\lambda = en$.

Imagine a spaceship flying parallel to the proton beam at the distance R with a velocity v.



b) What force F acts on the spacecraft when it carries a negative, point-like charge $-Q$? What direction has the force?

Now we look at the same situation from the perspective of the spaceship. We set here $v = v_{pr} = c/3$.

c) Transform the electric and magnetic fields into the coordinate system K' of the spacecraft.

d) We can also look at the the proton beam from the perspective of the spaceship in order to calculate the fields. Show that the fields calculated in K' coordinate system agree with the results from the subtasks c).

e) Calculate the force F' acting on the spacecraft. Knowing that the acceleration transforms like $\gamma^2 a = a'$ show how the mass transforms. We are now demanding that the equation of motion $F = ma$ is invariant under Lorentz transformations. Hence, it also holds that $F' = m'a'$.