1. Lorentz Group (Oral)

In this exercise, we prove that the Lorentz group is a group as well as we analyze its properties. First, let us revise the definition of a group.

Definition: A group is a set \( G \) together with an operation \( \cdot \) (called the group law of \( G \)) that combines any two elements \( a \) and \( b \) to form another element, denoted by \( a \cdot b \). To qualify as a group, the set and the operation, \( (G, \cdot) \), must satisfy four requirements known as the group axioms:

1) **Closure**: For all \( a, b \in G \), the result of the operation \( a \cdot b \) is also an element of \( G \).

2) **Associativity**: For all \( a, b, c \in G \) the following relation is satisfied \( (a \cdot b) \cdot c = a \cdot (b \cdot c) \).

3) **Identity element**: There exists an element \( e \in G \), such that for every element \( a \in G \), the equality \( e \cdot a = a \cdot e = a \) holds. Such an element is unique, and thus called the identity element.

4) **Inverse element**: For each \( a \in G \), there exists an element \( b \in G \) such that \( a \cdot b = b \cdot a = e \), where \( e \) is the identity element.

In the lecture, for Lorentz transformations, we defined \( G \) as the set characterized by invariance of the metric tensor \( g \) of Minkowski spacetime, i.e.

\[
G := \{ \Lambda \in \mathbb{R}^{4x4} \mid \Lambda^t g \Lambda = g \}
\]  

(1)

and the group operation \( \cdot \) as the multiplication of matrices.

a) Show that the Lorentz group is a group.

b) Depending on \( \det(\Lambda) \) and \( \text{sign}(\Lambda^0_0) \) the Lorentz group can be divided into four components. Show that proper orthochronous Lorentz transformations, i.e. \( \det(\Lambda) = 1 \) and \( \text{sign}(\Lambda^0_0) = 1 \), form a group.

c) Next, proof that each of three other components does not form a group. Give an example of the combination of two out of four components which again gives a proper group.

2. Galilean invariance of handicapped Maxwell equations (Oral)

In the lecture we learned that Maxwell’s equations are not invariant under Galilean transformations \( \mathbf{r}' = \mathbf{r} + v t, \ t' = t \). Here, we will show that Maxwell’s equations
become Galilean invariant if one drops the induction term in the Maxwell-Faraday equation

\begin{align*}
\nabla \cdot \mathbf{E} &= \rho / \epsilon_0, \\
\nabla \times \mathbf{E} &= -\partial / \partial t \mathbf{B}, \\
\nabla \cdot \mathbf{B} &= 0, \\
\nabla \times \mathbf{B} &= \mu_0 (j + \epsilon_0 \partial \mathbf{E} / \partial t).
\end{align*}

For this purpose show that above equations are form invariant if we define \( \mathbf{E}', \mathbf{B}' \) by

\[ \mathbf{E}' = \mathbf{E}, \quad \mathbf{B}' = \mathbf{B} + \epsilon_0 \mu_0 v \times \mathbf{E}. \]

*Hint:* You might find the following relations useful:

\begin{align*}
\nabla \cdot (\mathbf{a} \times \mathbf{b}) &= \mathbf{b} \cdot \nabla \times \mathbf{a} - \mathbf{a} \cdot \nabla \times \mathbf{b}, \\
\nabla \times (\mathbf{a} \times \mathbf{b}) &= \mathbf{a} \nabla \cdot \mathbf{b} - \mathbf{b} \nabla \cdot \mathbf{a} + (\mathbf{b} \cdot \nabla) \mathbf{a} - (\mathbf{a} \cdot \nabla) \mathbf{b}.
\end{align*}

3. Tractor beam (Oral)

A beam of protons flies along the \( x \)-axis, wherein the protons have a speed \( v_{pr} = c/3 \) and a density \( n \) (in protons per unit length).

a) Use the laws of electrostatics and magnetostatics to calculate the fields \( \mathbf{E}(r) \) and \( \mathbf{B}(r) \) generated by the beam as a function of the distance to the beam \( r \).

*Tip:* The fields produced by the beam are given by \( I = en v_{pr} \) and the charge distribution can be approximated for \( r \gg 1/n \) as continuous, what leads to the charge density \( \lambda = en \).

Imagine a spaceship flying parallel to the proton beam at the distance \( R \) with a velocity \( v \).
b) What force $F$ acts on the spacecraft when it carries a negative, point-like charge $-Q$? What direction has the force?

Now we look at the same situation from the perspective of the spaceship. We set here $v = v_{pr} = c/3$.

c) Transform the electric and magnetic fields into the coordinate system $K'$ of the spacecraft.

d) We can also look at the proton beam from the perspective of the spaceship in order to calculate the fields. Show that the fields calculated in $K'$ coordinate system agree with the results from the subtasks c).

e) Calculate the force $F'$ acting on the spacecraft. Knowing that the acceleration transforms like $\gamma^2 a = a'$ show how the mass transforms. We are now demanding that the equation of motion $F = ma$ is invariant under Lorentz transformations. Hence, it also holds that $F' = m'a'$.  

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