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You can find detailed information about the lecture and the exercises on the website `http://www.itp3.uni-stuttgart.de/lehre/vorlesungen/QFT.html`. Exercises are divided into two different groups. *Written* exercises have to be handed in and will be graded by the tutors. *Oral* exercises have to be prepared for the exercise session and will be presented by one of the students. In order to be admitted to the exam, we require (a) 80% of the written exercises to be solved or treated adequately, (b) 66% of the oral exercises to be prepared and (c) two exercises to be presented at the black board.

The first exercise sheet serves as a repetition for some important concepts from classical field theory.

Exercise 1: Functional derivative (Oral)

For a given manifold $\mathcal{M}$ of functions ϕ and a functional F with $F:\ \mathcal{M} \to \mathbb{R}$ or $\mathbb{C}$, the functional derivative $\frac{\delta F[\phi]}{\delta \phi}$ is defined as

$$\int \mathrm{d}x' \, \frac{\delta F[\phi]}{\delta \phi(x')} f(x') = \lim_{\varepsilon \to 0} \frac{F[\phi(x) + \varepsilon f(x)] - F[\phi(x)]}{\varepsilon} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} F[\phi + \varepsilon f] \bigg|_{\varepsilon = 0} \tag{1}$$

for all test functions $f \in \mathcal{M}$.

a) Show that for two functionals F and G

$$\frac{\delta (F + \lambda G)[\phi]}{\delta \phi(x)} = \frac{\delta F[\phi]}{\delta \phi(x)} + \lambda \frac{\delta G[\phi]}{\delta \phi(x)} \quad \text{for } \lambda \in \mathbb{R} \tag{2}$$

$$\frac{\delta (FG)[\phi]}{\delta \phi(x)} = \frac{\delta F[\phi]}{\delta \phi(x)} G[\phi] + F[\phi] \frac{\delta G[\phi]}{\delta \phi(x)}, \text{ with } (FG)[\phi] = F[\phi]G[\phi]. \tag{3}$$

If $G[\phi]$ is a function-like functional, i.e. can be treated as a function itself

$$\frac{\delta F[G[\phi]]}{\delta \phi(y)} = \int \mathrm{d}x \, \frac{\delta F[G]}{\delta G(x)} \frac{\delta G[\phi](x)}{\delta \phi(y)}. \tag{4}$$

b) Calculate the functional derivative $\frac{\delta F[\phi]}{\delta \phi(x)}$ for the following functionals:

$$F[\phi] = \int \mathrm{d}x' \, K(y, x') \phi(x'), \quad \text{where } K(y,x') \text{ is a so called \textit{integral kernel}} \tag{5}$$

$$F[\phi] = \phi(y) \tag{6}$$

$$F[\phi] = \phi'(y) \tag{7}$$

$$F[\phi] = \int \mathrm{d}y \, f(\phi(y), \phi'(y)) \ \text{ for a differentiable function } f:\ \mathbb{R} \times \mathbb{R} \to \mathbb{R} \tag{8}$$

Exercise 2: Lorentz covariance (Oral)

This exercise should serve as a **brief** revision of Lorentz covariance and the covariant formulation of classical electromagnetism. In the following, we will work in units where $c = 1$. Further, we will make use of *Einstein notation* where summation over indices appearing twice is assumed.

We first introduce the four-vector

$$x^\mu = (t, \mathbf{r}), \quad \mu = 0, 1, 2, 3, \tag{9}$$

which we will call a **contravariant** vector or tensor of first order. The vector x_μ is called **covariant** vector. In special relativity, the metric tensor is given by

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{10}$$

and the relationship between co- and contravariant vectors is given by

$$x_\mu = g_{\mu\nu} x^\nu. \tag{11}$$

A **Lorentz vector** is an object that under a Lorentz transformation $\Lambda^\mu{}_\nu$ transforms like

$$\tilde{x}^\mu = \Lambda^\mu{}_\nu x^\nu. \tag{12}$$

In tensors of higher order, each index transforms as a Lorentz vector, e.g.

$$\tilde{A}^{\alpha\beta\gamma}{}_{\delta\varepsilon} = \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu \Lambda^\gamma{}_\xi \Lambda_\delta{}^\rho \Lambda_\varepsilon{}^\sigma A^{\mu\nu\xi}{}_{\rho\sigma}. \tag{13}$$

A **Lorentz scalar** is a quantity that is invariant under Lorentz transformations.

a) Show that $x^\mu x_\mu$ is a Lorentz scalar, i.e. show that $x^\mu x_\mu = \tilde{x}^\sigma \tilde{x}_\sigma$.

Another important object is the four-gradient

$$\frac{\partial}{\partial x^\mu} = \partial_\mu = (\partial_t, \nabla). \tag{14}$$

b) Compute the d'Alembert operator $\partial^\mu \partial_\mu$. Is this quantity a Lorentz scalar?

In a covariant formulation of electromagnetism, the electric and magnetic field $\mathbf{E}$ and $\mathbf{B}$, respectively, are given by the anti-symmetric field tensor

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \qquad F^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{15}$$

The fields can also be described by the four-potential $A_\mu = (\Phi, -\mathbf{A})$, where Φ is a scalar potential and $\mathbf{A}$ is a vector potential.

c) Show that $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ reproduces the fields $\mathbf{E}$ and $\mathbf{B}$.

d) Show that $F_{\mu\nu}$ is invariant under the gauge transformation $A_\mu \to A_\mu - \partial_\mu f$, where f is an arbitrary function.

e) Show that in Lorenz gauge, $\partial_\nu A^\nu = 0$, and for no external sources, the Maxwell equations $\partial_\mu F^{\mu\nu} = 0$ reduce to $\partial^\mu \partial_\mu A^\nu = 0$.

Exercise 3: Maxwell equations (Written, 4 points)

The Maxwell equations for classical electromagnetism in vacuum can be derived from the action

$$S = \int \mathrm{d}^4x \mathcal{L} = \int \mathrm{d}^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}\right), \tag{16}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

a) Derive the Maxwell equations from the action (16). Use the Euler-Lagrange equations and treat the components $A_\mu(x)$ as the dynamic variables. Write the two 'inhomogeneous' Maxwell equations in their standard form by using $E^i = -F^{0i}$ and $\varepsilon^{ijk} B^k = -F^{ij}$, $i = x, y, z$. What happens with the two homogeneous equations?

b) Calculate the energy-momentum tensor $T^{\mu\nu}$ for the electromagnetic field

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\lambda)} \partial^\nu A_\lambda - g^{\mu\nu} \mathcal{L}. \tag{17}$$

c) This tensor, however, is not symmetric which can be fixed by adding a term of the form $\partial_\lambda K^{\lambda\mu\nu}$, where $K^{\lambda\mu\nu}$ is antisymmetric in its first two indices. Calculate the symmetric energy-momentum tensor

$$\hat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu} \tag{18}$$

with

$$K^{\lambda\mu\nu} = F^{\mu\lambda} A^\nu. \tag{19}$$

d) Show that the symmetrized tensor yields the standard form of the electromagnetic energy density and the momentum density (Poynting vector)

$$\mathcal{E} = \frac{1}{2}(E^2 + B^2) \quad \text{and} \quad \mathbf{S} = \mathbf{E} \times \mathbf{B}. \tag{20}$$