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April 12th, 2016
SS 2016

Exercise 1: Correlation and Commutation functions (Oral)

Evaluate the correlation function

$$\langle 0|\phi(x)\phi(y)|0\rangle = D(x-y) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{-ip\cdot(x-y)} \tag{1}$$

for $(x-y)$ space-like so that $(x-y)^2 = -r^2$

a) Express this correlation function in terms of Bessel functions and discuss its behavior for large separations.
(**Hint:**

$$K_0 = \int_0^\infty \frac{\cos axdx}{\sqrt{\beta^2+x^2}}, \tag{2}$$

$$-\beta\frac{\partial K_0(a\beta)}{\partial a} = \beta K_1(a\beta), \tag{3}$$

here, K_0 and K_1 are zero and first-order modified Bessel functions of the second kind, respectively.)

b) Calculate the commutator in space as well

$$\langle 0|\left[\phi(x),\phi(y)\right]|0\rangle \tag{4}$$

c) From the conclusions above, give the physical meanings in terms of causality.

Exercise 2: The complex scalar field (Written, 4 points)

Consider the field theory of a free complex scalar field. The action is given

$$S = \int \mathrm{d}^4x\,\mathcal{L} = \int \mathrm{d}^4x\,\left(\partial_\mu\phi^\dagger\partial^\mu\phi - m^2\phi^\dagger\phi\right). \tag{5}$$

We consider ϕ and $\phi^\dagger$ as the basic dynamical variables.

a) Find the conjugate momenta to ϕ and $\phi^\dagger$ and their canonical commutation relations. Show that the Hamiltonian is given by

$$H = \int \mathrm{d}^3x\,\left(\pi^\dagger\pi + \nabla\phi^\dagger\nabla\phi + m^2\phi\right) \tag{6}$$

b) Compute the Heisenberg equation of motion for ϕ and show that this yields the Klein-Gordon equation.

c) Introduce two different types of creation and annihilation operators $a_{\mathbf{k}}^{(\dagger)}$ and $b_{\mathbf{k}}^{(\dagger)}$ with

$$\left[a_{\mathbf{k}}, a_{\mathbf{k}'}^{\dagger}\right] = (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}') = \left[b_{\mathbf{k}}, b_{\mathbf{k}'}^{\dagger}\right] \tag{7}$$

and all other commutators equal to zero. Show that an expansion of ϕ into Fourier modes

$$\phi(x) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3 \sqrt{2E_{\mathbf{k}}}} \left(a_{\mathbf{k}} e^{-ikx} + b_{\mathbf{k}}^{\dagger} e^{ikx}\right) \tag{8}$$

diagonalizes the Hamiltonian. What is the difference compared to the theory of the real scalar (Klein-Gordon) field?

d) The Lagrangian in (5) is invariant under the symmetry $\phi \to e^{i\chi}\phi$, $(\chi \in \mathbb{R})$, which implies a conserved current according to the Noether theorem. Derive the conserved current j^μ and calculate the (conserved) charge $Q = \int \mathrm{d}^3 x\, j^0$. Interpret your result.