Exercise 1: Correlation and Commutation functions (Oral)

Evaluate the correlation function
\[ \langle 0 | \phi(x) \phi(y) | 0 \rangle = D(x - y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip \cdot (x-y)} \]  

for \((x - y)\) space-like so that \((x - y)^2 = -r^2\)

a) Express this correlation function in terms of Bessel functions and discuss its behavior for large separations.

(Hint:)

\[ K_0 = \int_0^\infty \frac{\cos ax dx}{\sqrt{\beta^2 + x^2}}; \]
\[ -\beta \frac{\partial K_0(a\beta)}{\partial a} = \beta K_1(a\beta), \]

here, \(K_0\) and \(K_1\) are zero and first-order modified Bessel functions of the second kind, respectively.)

b) Calculate the commutator in space as well
\[ \langle 0 | [\phi(x), \phi(y)] | 0 \rangle \]

c) From the conclusions above, give the physical meanings in terms of causality.

Exercise 2: The complex scalar field (Written, 4 points)

Consider the field theory of a free complex scalar field. The action is given
\[ S = \int d^4x \mathcal{L} = \int d^4x \left( \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi \right). \]

We consider \(\phi\) and \(\phi^\dagger\) as the basic dynamical variables.

a) Find the conjugate momenta to \(\phi\) and \(\phi^\dagger\) and their canonical commutation relations.

Show that the Hamiltonian is given by
\[ H = \int d^3x \left( \pi^\dagger \pi + \nabla \phi^\dagger \nabla \phi + m^2 \phi \right) \]

b) Compute the Heisenberg equation of motion for \(\phi\) and show that this yields the Klein-Gordon equation.
c) Introduce two different types of creation and annihilation operators $a_k^{(†)}$ and $b_k^{(†)}$ with

$$[a_k, a_{k'}^{†}] = (2\pi)^3 \delta^{(3)}(k - k') = [b_k, b_{k'}^{†}]$$

and all other commutators equal to zero. Show that an expansion of $\phi$ into Fourier modes

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 \sqrt{2E_k}} \left( a_k e^{-ikx} + b_k^{†} e^{ikx} \right)$$

diagonalizes the Hamiltonian. What is the difference compared to the theory of the real scalar (Klein-Gordon) field?

d) The Lagrangian in (5) is invariant under the symmetry $\phi \rightarrow e^{i\chi} \phi$, ($\chi \in \mathbb{R}$), which implies a conserved current according to the Noether theorem. Derive the conserved current $j^\mu$ and calculate the (conserved) charge $Q = \int d^3x \ j^0$. Interpret your result.