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Exercise 1: Feynman propagator (Oral)

We consider the two field operators $\phi^\dagger(t, \mathbf{x})$ and $\phi(t', \mathbf{x}')$. If $t' > t$, we first create a particle at time t with $\phi^\dagger(t, \mathbf{x})$ and destroy it later with $\phi(t', \mathbf{x}')$. The amplitude will be given by the expectation value of

$$\theta(t' - t)\phi(t', \mathbf{x}')\phi^\dagger(t, \mathbf{x}). \tag{1}$$

If $t > t'$, we first create an antiparticle at time t' with $\phi(t', \mathbf{x}')$ and destroy it later with $\phi^\dagger(t, \mathbf{x})$. The amplitude will be given by the expectation value of

$$\theta(t - t')\phi^\dagger(t, \mathbf{x})\phi(t', \mathbf{x}'). \tag{2}$$

With the combination of these two terms, we can define the Feynman propagator $D_F(t, t', \mathbf{x}, \mathbf{x}')$ with time-ordering operator as

$$T\phi(t', \mathbf{x}')\phi^\dagger(t, \mathbf{x}) = \theta(t' - t)\phi(t', \mathbf{x}')\phi^\dagger(t, \mathbf{x}) + \theta(t - t')\phi^\dagger(t, \mathbf{x})\phi(t', \mathbf{x}'). \tag{3}$$

a) Please calculate the first and second derivative of $T\phi(t', \mathbf{x}')\phi^\dagger(t, \mathbf{x})$ with respect to t'.

b) Please prove that

$$\left(\partial_{\mu'}\partial^{\mu'} + m^2\right) T\phi(t', \mathbf{x}')\phi^\dagger(t, \mathbf{x}) = -i\delta^{(4)}(x' - x). \tag{4}$$

Exercise 2: Fock states and coherent states (Written, 5 points)

In the lecture it was discussed that the single particle states $|\mathbf{p}\rangle = \sqrt{2E_\mathbf{p}} a^\dagger_\mathbf{p}|0\rangle$ are not well suited to build up a quantum field theory since the diverging commutation relations among $a_\mathbf{p}$ and $a^\dagger_\mathbf{p}$ prevents the state from being normalized.

Let us define instead the operator

$$a^\dagger(f) \equiv \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2E_\mathbf{p}}}\, f(\mathbf{p})\, a^\dagger_\mathbf{p}, \tag{5}$$

which creates a particle in the f mode.

a) Calculate the commutator $[a(f), a^\dagger(f)]$. What property must $f(\mathbf{p})$ satisfy in order for the states to be normalizable?

b) Consider now a generalization to n particles. We may define the unnormalized *Fock state* as

$$|n\rangle = \int \frac{\mathrm{d}^3 p_1}{(2\pi)^3 \sqrt{2E_{\mathbf{p}_1}}} \cdots \int \frac{\mathrm{d}^3 p_n}{(2\pi)^3 \sqrt{2E_{\mathbf{p}_n}}} F(\mathbf{p}_1, \ldots, \mathbf{p}_n)\, a^\dagger_{\mathbf{p}_1} \cdots a^\dagger_{\mathbf{p}_n} |0\rangle, \tag{6}$$

where $F(\mathbf{p}_1, \ldots, \mathbf{p}_n)$ is symmetric under the exchange of two of its arguments. Calculate the norm of (6). Show that (6) is an eigenstate of the number operator

$$N = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} a^\dagger_{\mathbf{p}} a_{\mathbf{p}} \tag{7}$$

and calculate its eigenvalue.

c) Calculate the expectation value of the (normal ordered) Hamiltonian

$$H = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} E_{\mathbf{p}} a^\dagger_{\mathbf{p}} a_{\mathbf{p}} \tag{8}$$

of the real Klein-Gordon field with respect to the state (6).

d) Consider now a coherent superposition of n-particle states, i.e. a *coherent state*,

$$|\alpha\rangle = \exp\left(\int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2E_{\mathbf{p}}}} \alpha(\mathbf{p})\, a^\dagger_{\mathbf{p}} \right) |0\rangle \tag{9}$$

Calculate its norm as well as the overlap of two (normalized) coherent states, i.e. $\langle \alpha | \beta \rangle$. What does this result imply?

e) Show that (9) is an eigenstate of the annihilation operator $a_{\mathbf{p}}$ and calculate its eigenvalue. Show that the coherent state remains coherent under time evolution with the Hamiltonian (8), i.e. show that $e^{-iHt}|\alpha\rangle = |\beta\rangle$, where $|\beta\rangle$ has to be determined.