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April 26th, 2016
SS 2016

Exercise 1: Plane Wave Solutions for the Dirac Equation (Oral)

In this exercise we will study in detail the Dirac equation and its solutions in the Weyl or chiral representation.

Let us start by writing the Dirac equation for a relativistic massive particle, $m \neq 0$, with wave function ψ,

$$(i\gamma^\mu \partial_\mu - m)\psi = 0, \tag{1}$$

Notice that the Klein-Gordon equation is just obtained by multiplying by $(i\gamma^\mu \partial_\mu + m)$. We shall choose the *Weyl* (also called *chiral*) representation for the generators of the Dirac algebra,

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{2}$$

where I and σ^i stand for the 2×2 unit matrix and the Pauli matrices, respectively. In this four-dimensional representation, the wave function ψ may be written as a bispinor

$$\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix} \tag{3}$$

where ϕ and χ are two-component spinors. We seek plane wave solutions of (1), i.e. solutions of the form

$$\begin{aligned} \psi^{(+)}(x) &= e^{-ik\cdot x} u(k) \quad \text{positive energy} \\ \psi^{(-)}(x) &= e^{ik\cdot x} v(k) \quad \text{negative energy}, \end{aligned} \tag{4}$$

with the condition that $k^0 > 0$. To verify the Klein-Gordon equation, we also must have $k^2 = m^2$. In the following, we will express all quantities in units of $\hbar = 1$.

a) Starting from the Dirac equation in the chiral representation, write the equations that $u(k)$ and $v(k)$ must satisfy.

b) Write the reduced equations for $u(m, \mathbf{0})$, and $v(m, \mathbf{0})$ in the rest frame of the particle, i.e. $k^\mu = (m, \mathbf{0})$.

c) There are clearly two linearly independent solutions for u and two for v. In the chiral representation (2), write the solutions for $u^{(\alpha)}(m, \mathbf{0})$ and $v^{(\alpha)}(m, \mathbf{0})$, where α is the index of two linearly independent solutions.

d) We shall write the complete solution as

$$\begin{aligned} u^{(\alpha)}(k) &= U^{(\alpha)}(k)u^{(\alpha)}(m,\mathbf{0}), \quad (5)\\ v^{(\alpha)}(k) &= V^{(\alpha)}(k)v^{(\alpha)}(m,\mathbf{0}), \quad (6)\end{aligned}$$

show what are the equations that $U^{(\alpha)}(k)$ and $V^{(\alpha)}(k)$ must satisfy.

e) We shall define the conjugate wave function as

$$\bar{\psi} \equiv \psi^\dagger \gamma^0, \quad (7)$$

write the equations that $\bar{u}^\alpha(k)$ and $\bar{v}^\alpha(k)$ must satisfy.

f) In evaluating Feynman diagrams, we will often wish to sum over the polarization states of fermion. We can derive the relevant completeness relations with the following sums:

$$\begin{aligned} \Lambda_+(k) &\equiv \sum_{\alpha=1,2} u^\alpha(k)\bar{u}^\alpha(k)\\ \Lambda_-(k) &\equiv -\sum_{\alpha=1,2} v^\alpha(k)\bar{v}^\alpha(k). \quad (8)\end{aligned}$$

please calculate them.

Exercise 2: Relativistic Hydrogen-like atoms (Written, 5 points)

In this exercise, we want to calculate the spectrum of the relativistic hydrogen atom with the Dirac equation (therefore including spin). To this end, we consider the coupling to an external electromagnetic field characterized by its potential A_μ via the minimal coupling prescription

$$\partial_\mu \to \partial_\mu + ieA_\mu, \quad (9)$$

where e is the (negative) electric charge. In quantum electrodynamics, the electric charge is considered as a coupling constant and in natural units is dimensionless. It is related to the fine-structure constant (also known as Sommerfeld's constant) by $\alpha = \frac{e^2}{4\pi}$.

a) Consider the Klein-Gordon equation with minimal coupling and a static vector potential $A_0 = \frac{Ze}{4\pi r}$. Use the ansatz $\phi(t,\mathbf{r}) = e^{-iEt}\phi(\mathbf{r})$ and show that the Klein-Gordon equation in spherical coordinates reduces to

$$\left(-\partial_r^2 - \frac{2}{r}\partial_r + \frac{L^2 - Z^2\alpha^2}{r^2} - \frac{2Z\alpha E}{r} - (E^2 - m^2)\right)\phi = 0, \quad (10)$$

with L^2 the angular momentum operator.

b) Use the substitutions

$$L^2 \rightarrow L^2 - Z^2\alpha^2 \tag{11}$$

$$\alpha \rightarrow \alpha \frac{E}{m} \tag{12}$$

$$\varepsilon \rightarrow \frac{E^2 - m^2}{2m} \tag{13}$$

to get an equation which is equivalent to the Schrödinger equation for the non-relativistic hydrogen atom. Show that the spectrum of the relativistic hydrogen atom without spin is then given by

$$E_{nl} = \frac{m}{\sqrt{1 + (Z^2\alpha^2/(n - \delta_l)^2)}}, \tag{14}$$

with

$$\delta_l = l + \frac{1}{2} - \sqrt{\left(l + \frac{1}{2}\right)^2 - Z^2\alpha^2}. \tag{15}$$

c) In order to include the spin, we turn to the Dirac equation. Write down the Dirac equation with the minimal coupling prescription (9). Multiply the equation by $i\gamma^\mu\partial_\mu - e\gamma^\mu A_\mu + m$ and bring your result into the form

$$\left[(i\partial_\mu - eA_\mu)^2 - \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu} - m^2\right]\psi = 0, \tag{16}$$

with

$$\sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right]. \tag{17}$$

We choose for the four-potential A_μ again $A_0 = -\frac{Ze}{4\pi r}$, $\mathbf{A} = 0$. Using the Weyl representation for the γ matrices, show that (16) reduces to

$$\left[-\left(\partial_r^2 + \frac{2}{r}\partial_r\right) + \frac{L^2 - Z^2\alpha^2 \mp iZ\alpha\boldsymbol{\sigma}\cdot\hat{r}}{r^2} - \frac{2Z\alpha E}{r} - (E^2 - m^2)\right]\psi_\pm = 0, \tag{18}$$

where $\hat{r} = \mathbf{r}/r$.

d) In order to diagonalize the Hamiltonian in (18), we introduce the total angular momentum operator $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{\sigma}/2$ which commutes with the Hamiltonian and L^2. Consider now the subspace where $J^2 = j(j+1)$, $J_z = m$ ($j = \frac{1}{2}, \frac{3}{2}, \ldots$; $-j \leq m \leq j$) and $L^2 = l(l+1)$. Which values the integer l can take? Show that in this subspace the operator $L^2 - Z^2\alpha^2 \mp iZ\alpha\boldsymbol{\sigma}\hat{r}$ takes the following form

$$L^2 - Z^2\alpha^2 \mp iZ\alpha\boldsymbol{\sigma}\hat{r} = \begin{pmatrix} (j+\frac{1}{2})(j+\frac{3}{2}) - Z^2\alpha^2 & \mp iZ\alpha \\ \mp iZ\alpha & (j-\frac{1}{2})(j+\frac{1}{2}) - Z^2\alpha^2 \end{pmatrix}. \tag{19}$$

Assume this matrix has eigenvalues $\lambda(\lambda+1)$ and show that they can be written

$$\lambda = \left(j \pm \frac{1}{2}\right) - \delta_j \tag{20}$$

with

$$\delta_j = j + \frac{1}{2} - \sqrt{\left(j + \frac{1}{2}\right)^2 - Z^2\alpha^2}. \tag{21}$$

Calculate the spectrum similar to the case of the Klein-Gordon field and show that the energies are given by

$$E_{nj} = \frac{m}{\sqrt{1 + (Z^2\alpha^2/(n-\delta_j)^2)}}. \tag{22}$$

e) Expand the energies E_{nl} and E_{nj} up to $\mathcal{O}(\alpha^4)$ and discuss the spectrum in both cases. What are the differences with respect to the non-relativistic spectrum?