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Insitute for Theoretical Physics III, University of Stuttgart May 3rd, 2016

Exercise 1: Transformation of Parity (Oral)

There are three types of action on Dirac particles and fields: parity (P), time (T) and charge (C). In this exercise, we focus on the transformation of parity P, which reverses the momentum of a particle without flipping its spin. Mathematically, this means that P should be implemented by a unitary operator which, for example, transforms the states $a^s_{\mathbf{p}}|0\rangle$ into $a^s_{-\mathbf{p}}|0\rangle$. In other words, we want

$$Pa^s_{\mathbf{p}}P = \eta_a a^s_{-\mathbf{p}}, \quad Pb^s_{\mathbf{p}}P = \eta_b b^s_{-\mathbf{p}} \tag{1}$$

where η_a and η_b are possible phases. Notice that first, two applications of the parity operator should return observables to their original values and second, observables are built from an *even* number of fermionic operators. Therefore,

$$\eta_a^2, \quad \eta_b^2 = 1. \tag{2}$$

We will follow the representation for γ matrices in Peskin's book.

a) Let us start from the field operator in the quantized Dirac field as

$$\begin{aligned}\psi(x) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\sum_s (a^s_{\mathbf{p}}u^s(p)e^{-ip\cdot x} + b^{s\dagger}_{\mathbf{p}}v^s(p)e^{ip\cdot x}),\\ \bar{\psi}(x) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\sum_s (b^s_{\mathbf{p}}\bar{v}^s(p)e^{-ip\cdot x} + a^{s\dagger}_{\mathbf{p}}\bar{u}^s(p)e^{ip\cdot x}).\end{aligned} \tag{3}$$

Here the coordinate x include the dimension of time (t, x). Please prove the following formulas

$$\begin{aligned}P\psi(t,\mathbf{x})P &= \eta_a\gamma^0\psi(t,-\mathbf{x}),\\ P\bar{\psi}(t,\mathbf{x})P &= \eta_a^*\bar{\psi}(t,-\mathbf{x})\gamma^0.\end{aligned} \tag{4}$$

b) Please calculate the transformation laws for the terms

$$P\bar{\psi}\psi P, \qquad P\bar{\psi}\gamma^\mu\psi P, \qquad Pi\bar{\psi}\gamma^5\psi P, \qquad P\bar{\psi}\gamma^\mu\gamma^5\psi P, \qquad P\bar{\psi}[\gamma^\mu,\gamma^\nu]\psi P. \tag{5}$$

Exercise 2: Weyl equations and Majorana fermions (Oral)

Recall from the lecture that the spinor representation of the Lorentz transformation Λ is given by

$$\Lambda_{\frac{1}{2}} = \exp\left(-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right), \tag{6}$$

where $\omega_{\mu\nu}$ is an anti-symmetric tensor and $S^{\mu\nu}$ are the generators for the boosts β and rotations θ,

$$S^{0i} = -\frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}, \qquad S^{ij} = \frac{1}{2}\varepsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}, \tag{7}$$

respectively, in the chiral representation. Due to the block-diagonal form of the generators, the so(1,3) or Dirac representation of the Lorentz group is reducible into two 2-dimensional representations, and the Dirac spinor can be written as

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \tag{8}$$

where ψ_L and ψ_R are the left- and right-handed *Weyl spinors*, respectively.

a) Show that under infinitesimal rotations $\boldsymbol{\theta}$ and boosts $\boldsymbol{\beta}$, the left- and right-hand Weyl spinors transform as

$$\psi_L \rightarrow \left(1 - i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2} - \boldsymbol{\beta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)\psi_L, \tag{9}$$

$$\psi_R \rightarrow \left(1 - i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2} + \boldsymbol{\beta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)\psi_R. \tag{10}$$

Show that the quantity $\sigma^2\psi_L^*$ transforms like a right-handed spinor.
(*Hint:* $\sigma^2\boldsymbol{\sigma}^* = -\boldsymbol{\sigma}\sigma^2$)

b) Introduce the notation

$$\sigma^\mu \equiv (1, \boldsymbol{\sigma}), \quad \bar{\sigma}^\mu \equiv (1, -\boldsymbol{\sigma}), \quad \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \tag{11}$$

and show for the case of massless fermions the Dirac equation in the Weyl representation decouples into the so-called *Weyl equations*

$$i\bar{\sigma}^\mu\partial_\mu\psi_L = 0, \qquad i\sigma^\mu\partial_\mu\psi_R = 0. \tag{12}$$

c) Show that it is possible to write an equation for a two-component spinor $\chi(x)$ as a massive field in the following way:

$$i\bar{\sigma}^\mu\partial_\mu\chi - im\sigma^2\chi^* = 0. \tag{13}$$

That is, show that this equation is relativistically invariant and that it implies the Klein-Gordon equation, $(\partial^2 + m^2)\chi = 0$. This form of the fermion mass is called a *Majorana mass term.*

d) We shall show now that the Majorana equation (13) can be obtained from a Lagrangian. To this end, we develop a classical theory for the field $\chi(x)$ by considering it as a classical anticommuting complex field, i.e. a *Grassmann field*, obeying the relations

$$\chi(x)\chi(x') = -\chi(x')\chi(x), \tag{14}$$

and where $\chi^\dagger = (\chi^*)^T$. Note that this relation implies $\chi(x)^2 = 0$. A Grassmann field $\chi(x)$ can be expanded

$$\chi(x) = \sum_n \alpha_n \phi_n(x), \tag{15}$$

where $\phi_n(x)$ are a set of orthogonal complex functions and the α_n are Grassmann numbers. We may define the complex conjugation of a product of two Grassmann numbers as

$$(\alpha\beta)^* \equiv \beta^*\alpha^* = -\alpha^*\beta^*, \tag{16}$$

so as to imitate the Hermitian conjugation of quantum fields. Show that the classical action,

$$S = \int \mathrm{d}^4x \left[\chi^\dagger i\bar{\sigma}^\mu \partial_\mu \chi + \frac{im}{2}\left(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*\right)\right], \tag{17}$$

is real, i.e. $S = S^*$, and that varying S with respect to χ and χ^* yields the Majorana equation (12).

e) Consider the Dirac field in the form given by (8) and recall that the lower component, i.e. ψ_R transforms equivalently to a unitary transformed ψ_L^* field. In this way, we can rewrite the 4-component Dirac field in terms of two 2-component Weyl spinors:

$$\psi_L(x) = \chi_1(x), \qquad \psi_R(x) = i\sigma^2\chi_2^*(x). \tag{18}$$

Rewrite the Dirac Lagrangian in terms of χ_1 and χ_2 and notice the form of the mass term.

f) Let us now quantize the Majorana theory (17) by the canonical quantization procedure. That is, let us promote $\chi(x)$ to a quantum field satisfying the canonical anticommutation relation

$$\left\{\chi_a(\mathbf{x}), \chi_b^\dagger(\mathbf{y})\right\} = \delta_{ab}\delta(\mathbf{x}-\mathbf{y}), \tag{19}$$

and find its conjugate momentum field to construct a Hermitian Hamiltonian. Find the set of fields that written in terms of annihilation and creation operators diagonalizes the Hamiltonian.
(Hint: Compare $\chi(x)$ to the two top components of the quantized Dirac field.)

Exercise 3: γ-matrices (Written, 3 points)

Consider a set of four matrices γ^μ satisfying the Dirac algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} I. \tag{20}$$

Further, we introduce a fifth matrix

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3. \tag{21}$$

a) Show without using any specific representation for the γ^μ that

$$\left(\gamma^5\right)^\dagger = \gamma^5, \qquad \left(\gamma^5\right)^2 = I, \qquad \left\{\gamma^5, \gamma^\mu\right\} = 0. \tag{22}$$

b) Prove without using any specific representation the following trace identities:

$$\mathrm{Tr}\left(\gamma^\mu\right) = 0, \quad \mathrm{Tr}\left(\gamma^\mu\gamma^\nu\right) = 4g^{\mu\nu}, \quad \mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) = 4\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right), \tag{23}$$

$$\mathrm{Tr}\left(\gamma^5\right) = 0, \qquad \mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^5\right) = 0, \tag{24}$$

$$\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5\right) = -4i\varepsilon^{\mu\nu\rho\sigma}, \qquad \mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\cdots\right) = \mathrm{Tr}\left(\cdots\gamma^\sigma\gamma^\rho\gamma^\nu\gamma^\mu\right). \tag{25}$$

c) Prove the following contraction identities without using any specific representation

$$\gamma^\mu\gamma_\mu = 4, \qquad \gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu, \tag{26}$$

$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4g^{\nu\rho}, \qquad \gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu. \tag{27}$$