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Exercise 1: Perturbation theory of the two-point correlation function (Written 6 points)

In this exercise, we will study the time-evolution operator, a key building block in perturbation theory. Our aim is to relate the two-point correlation function (Green's function) of an interacting theory,

$$\langle\Omega|\,\mathcal{T}\phi(x)\phi(y)\,|\Omega\rangle\,, \tag{1}$$

where $\mathcal{T}$ refers to the time-ordered product, and $|\Omega\rangle$ to the vacuum of the theory of interest, which we shall generically write as

$$H = H_0 + H_{int}(\lambda), \tag{2}$$

where H_0 is some exactly solvable (i.e. non-intaracting) field theory with $|0\rangle$ as its vacuum, and $H_{int}(\lambda)$ is some perturbation depending in a small parameter λ. For the present purpose, it will suffice to take H_0 to be the Klein-Gordon Hamiltonian and H_{int} some point-interaction. As will be seen later in the course, this two-point correlation function (1) plays a central role in quantum field theory as it appears in the computation of physical quantities such as scattering cross sections and decay rates.

Recall that in the Heisenberg picture any field operator at any time $t > t_0$ has the form

$$\phi(t,\mathrm{x}) = e^{iH(t-t_0)}\phi(t_0,\mathrm{x})e^{-iH(t-t_0)}. \tag{3}$$

For $\lambda = 0$, H becomes H_0 and this reduces to

$$\phi(t,\mathrm{x})|_{\lambda=0} = e^{iH_0(t-t_0)}\phi(t_0,\mathrm{x})e^{-iH_0(t-t_0)} \equiv \phi_I(t,\mathrm{x}). \tag{4}$$

When λ is small, this expression will still give the most important part of the time dependence of $\phi(x)$, and thus it is convenient to give this quantity a name: the interaction picture field, $\phi_I(t,\mathrm{x})$. We may extend such definition in the interaction picture to any operator Θ as

$$\Theta_I(t,\mathrm{x}) = e^{iH_0(t-t_0)}\Theta(t_0,\mathrm{x})e^{-iH_0(t-t_0)}. \tag{5}$$

a) We can write ϕ (Heisenberg picture) in terms of ϕ_I,

$$\begin{aligned}\phi(t,\mathrm{x}) &= e^{iH(t-t_0)}e^{-iH_0(t-t_0)}\phi_I(t,\mathrm{x})e^{iH_0(t-t_0)}e^{-iH(t-t_0)}\\ &\equiv U^\dagger(t,t_0)\phi_I(t,\mathrm{x})U(t,t_0),\end{aligned} \tag{6}$$

for which we have defined the *evolution operator*,

$$U(t,t_0) = e^{iH_0(t-t_0)}e^{-iH(t-t_0)}. \tag{7}$$

Prove that $U(t,t_0)$ satisfies the differential equation

$$i\frac{\partial}{\partial t}U(t,t_0) = H_I(t)U(t,t_0), \tag{8}$$

where we refer by $H_I(t)$ to H_{int} in the interaction picture.

b) Verify that $U(t,t_0)$ has as solution

$$\begin{aligned} U(t,t_0) = 1 \quad &+ \quad (i)\int_{t_0}^{t} dt_1 H_I(t_1) + (-i)^2\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 H_I(t_1)H_I(t_2) \\ &+ \quad (-i)^3\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_3} dt_3 H_I(t_1)H_I(t_2)H_I(t_3) + \cdots, \end{aligned} \tag{9}$$

which statisfies the Eq.(8).

c) By changing the areas for integration in Eq.(9), verify that this solution is equivalent to

$$U(t,t_0) = \mathcal{T}\{\exp[-i\int_{t_0}^{t} dt' H_I(t')]\}. \tag{10}$$

d) Verify that the evolution operator $U(t,t')$ satisfy the following properties

$$\begin{aligned} U^{-1} &= U^{\dagger}, & (11)\\ U(t_1,t_2)U(t_2,t_3) &= U(t_1,t_3), & (12) \end{aligned}$$

that together with the fact that $U(t',t') = 1$ provide it with a group structure.

e) Let us study the ground state $|\Omega\rangle$ of the whole interacting Hamiltonian. Imagine that we let evolve the ground-state of H_0, i.e. $|0\rangle$, with the complete Hamiltonian to some great future time T when the system is already thermalized,

$$\begin{aligned} e^{-iHT}|0\rangle &= \sum_n e^{iE_nT}|n\rangle\langle n|0\rangle, \\ e^{-iHT}|0\rangle &= e^{iE_0T}|\Omega\rangle\langle\Omega|0\rangle + \sum_{n\neq 0} e^{iE_nT}|n\rangle\langle n|0\rangle, \end{aligned} \tag{13}$$

here E_n are the eigenvalues of H. By the projection method, pushing T to some slightly imaginary infinity $T = \infty(1-i\epsilon)$, verify that

$$|\Omega\rangle = \lim_{T\longrightarrow\infty(1-i\epsilon)} (e^{-iE_0(t_0-(-T))}\langle\Omega|0\rangle)^{-1}U(t_0,-T)|0\rangle. \tag{14}$$

f) Finally, prove that the correlation function and that the Green's function have the form

$$\langle\Omega|\,\phi(x)\phi(y)\,|\Omega\rangle = \lim_{T\longrightarrow\infty(1-i\epsilon)} \frac{\langle 0|\,U(T,x^0)\phi_I(x)U(x^0,y^0)\phi_I(y)U(y^0,-T)\,|0\rangle}{\langle 0|\,U(T,-T)\,|0\rangle}, \quad (15)$$

and

$$\langle\Omega|\,\mathcal{T}\phi(x)\phi(y)\,|\Omega\rangle = \lim_{T\longrightarrow\infty(1-i\epsilon)} \frac{\langle 0|\,\mathcal{T}\{\phi_I(x)\phi_I(y)\exp[-i\int_{-T}^{T}dtH_I(t)]\}\,|0\rangle}{\langle 0|\,\mathcal{T}\{\exp[-i\int_{-T}^{T}dtH_I(t)]\}\,|0\rangle}, \quad (16)$$

respectively.

Exercise 2: Wick theorem (Oral)

In this exercise, we will derive a generating function with which we can prove the Wick theorem.

a) First, consider two operators A and B with $[A,[A,B]] = [B,[A,B]] = 0$. Prove the Baker-Campbell-Hausdorff formula

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]}. \quad (17)$$

Hint: Consider $f(t) = e^{tA}e^{tB}$ and derive a differential equation for $f(t)$ which you can solve. For $t = 1$, you will recover (17).

b) Consider the operator

$$S = U(-\infty,\infty) = \mathcal{T}\exp\left(-i\int_{-\infty}^{+\infty}\mathrm{d}t\,H_I(t)\right) = \lim_{\substack{t_i\to-\infty\\ t_f\to+\infty}} \mathcal{T}\exp\left(-i\int_{t_i}^{t_f}\mathrm{d}t\,H_I(t)\right),$$

where the commutator of H_I for different times is a c-number. Show that this operator can be rewritten as

$$S = \exp\left(-i\int \mathrm{d}tH_I(t)\right)\exp\left(-\frac{1}{2}\int\int \mathrm{d}t\,\mathrm{d}t'\,\theta(t-t')\,[H_I(t),H_I(t')]\right). \quad (18)$$

Hint: Discretize the interval $[t_i,t_f]$ and rewrite the time-ordered exponential as a product of exponentials.

c) Use

$$H_I(t) = \int \mathrm{d}^3x\,\phi(x)j(x), \quad (19)$$

where $\phi(x)$ is the real scalar field and $j(x)$ is some source term it is coupled to and show that

$$S = \,:e^{-i\int \mathrm{d}^4x\phi(x)j(x)}:\, e^{\frac{1}{2}\int\int \mathrm{d}^4x\mathrm{d}^4y\, j(x)\left\{[\phi^{(-)}(x),\phi^{(+)}(y)]-\theta(x^0-y^0)[\phi(x),\phi(y)]\right\}j(y)}. \quad (20)$$

Here, $\phi(x) = \phi^{(+)}(x) + \phi^{(-)}(x)$, where $\phi^{(+)}$ contains all the annihilation operators and $\phi^{(-)}$ all the creation operators.
Hint: $:e^{a+a^\dagger}: = e^{a^\dagger}e^a$

d) Using the fact that the commutator for free fields is just a c-number, show that (20) can be further simplified to

$$S = :e^{-i\int \mathrm{d}^4x\phi(x)j(x)}:\, e^{-\frac{1}{2}\int\int \mathrm{d}^4x\mathrm{d}^4y\, j(x)j(y)\langle 0|\mathcal{T}\phi(x)\phi(y)|0\rangle}. \tag{21}$$

e) Use (21) to prove the *Wick theorem*

$$\mathcal{T}\phi(x_1)\phi(x_2)\cdots\phi(x_m) = :\phi(x_1)\phi(x_2)\cdots\phi(x_m) + \text{all possible contractions}:. \tag{22}$$

The *contraction* of two fields is defined as

$$\overbrace{\phi(x)\phi}(y) = \begin{cases} [\phi^{(+)}(x), \phi^{(-)}(y)] & \text{for } x^0 > y^0, \\ [\phi^{(+)}(y), \phi^{(-)}(x)] & \text{for } y^0 > x^0. \end{cases} \tag{23}$$

Hint:

$$\overbrace{\phi(x)\phi}(y) = D_F(x-y) \tag{24}$$