Exercise 1: Cross Section of Two Scattering Particles (Written, 4 points)

In this exercise, we will study the cross section of two particles with $\phi^4$ theory. We start from the general formula for the differential cross section which includes two in-going particles and $n$ out-going particles,

$$d\sigma_{n\rightarrow 2} = \frac{1}{4[(p_1 \cdot p_2)^2 - m_1^2 m_2^2]^{1/2}} \int \Delta d\tilde{p}_3 \cdots d\tilde{p}_{n+2} |\langle p_3, \cdots, p_{n+2} | T | p_1, p_2 \rangle|^2 \times (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - \cdots - p_{n+2}).$$

(1)

a) Show that in $\phi^4$-theory, up to lowest order, the differential cross section in the center-of-mass frame is given by

$$\left( \frac{d\sigma}{d\Omega} \right) \quad \frac{\lambda^2}{64\pi^2 E_{cm}^2},$$

(2)

where $E_{cm}$ is the center-of-mass energy.

b) Calculate the total cross section.

Exercise 2: Rutherford scattering (Oral)

The cross section for the scattering of an electron by the Coulomb field of a nucleus can be computed, to lowest order, without quantizing the electromagnetic field. Instead, let us treat the field as a given classical potential $A_\mu(x)$. The interaction Hamiltonian is

$$H_I = \int d^3x \bar{\psi}(x) \gamma^\mu \psi(x) A_\mu(x),$$

(3)

where $\psi(x)$ is the usual quantized Dirac field.

a) Show that the $T$-matrix element for electron scattering off a localized classical potential is, to lowest order,

$$\langle p' | iT | p \rangle = -ie\bar{u}(p') \gamma^\mu u(p) \cdot \tilde{A}_\mu(p' - p),$$

(4)

where $\tilde{A}_\mu(q) = \int d^4x e^{-iqx} A_\mu(x)$ is the four-dimensional Fourier transform of $A_\mu(x)$.

b) If $\tilde{A}_\mu(q)$ is time independent, its Fourier transform contains a delta function of energy. It is then natural to define

$$\langle p' | iT | p \rangle \equiv -i\mathcal{M} \cdot (2\pi) \delta(E_f - E_i),$$

(5)
where $E_i$ and $E_f$ are the initial and final energies of the particle, and to adopt a new Feynman rule for computing $\mathcal{M}$:

$$\mathcal{M} = -ie\gamma^{\mu}\tilde{A}_\mu(q)$$  \hspace{1cm} (6)

where $\tilde{A}(q)$ is the three-dimensional Fourier transform of $A_\mu(x)$. Given this definition of $\mathcal{M}$, show that the cross section for scattering off a time-dependent localized potential is

$$d\sigma = \frac{1}{v_i} \frac{1}{2E_i} \frac{d^3p_f}{(2\pi)^3} \frac{1}{2E_f} |\mathcal{M}(p_i \to p_f)|^2 (2\pi)\delta(E_f - E_i),$$  \hspace{1cm} (7)

where $v_i$ is the particle’s initial velocity. Integrate over $|p_f|$ to find a simple expression for $d\sigma/d\Omega$.

c) Let us restrict to the case of electron scattering off a Coulomb potential

$$A^0 = \frac{Ze}{4\pi r}.$$  \hspace{1cm} (8)

Working in the non-relativistic limit, derive the Rutherford formula,

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 Z^2}{4m^2v^4 \sin^4(\theta/2)},$$  \hspace{1cm} (9)