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Insitute for Theoretical Physics III, University of Stuttgart June 7th, 2016

Exercise 1: Yukawa theory (Oral)

In this exercise, we study the Yukawa theory, which is a simplified model of Quantum Electodynamics. For definiteness, we begin by analyzing the Yukawa theory:

$$H = H_{\text{Dirac}} + H_{\text{Klein-Gordon}} + \int d^3x g\bar{\psi}\psi\phi. \tag{1}$$

To be even more specific, consider the two-particle scatering reaction

$$\text{fermion}(p) + \text{fermion}(k) \longrightarrow \text{fermion}(p') + \text{fermion}(k'). \tag{2}$$

The leading contribution comes from the H_I^2 term of S-matrix:

$$_0\langle \mathbf{p}', \mathbf{k}'| \mathcal{T}\left(\frac{1}{2!}(-ig)\int d^4x\, \bar{\psi}_I\psi_I\phi_I(-ig)\int d^4y\, \bar{\psi}_I\psi_I\phi_I\right)|\mathbf{p},\mathbf{k}\rangle_0 . \tag{3}$$

To evaluate this expression, use Wick's theorem to reduce the $\mathcal{T}$-product to an N-product of contractions, then act the uncontracted fields on the initial- and final- state particles.

a) Calculate the state after applying field operators to initial and final state,

$$\overbrace{\psi_I(x)\,|\mathbf{p}}, s\rangle \quad \text{and} \quad \langle s', \overbrace{\mathbf{p}'|\,\bar{\psi}_I}(x), \tag{4}$$

here just consider only one particle both in initial- and final-state, $\mathbf{p}$, $\mathbf{p}'$ are momentum and s, s' are spin polarization.

b) Prove the following process

$$\langle \mathbf{p}', \mathbf{k}'| \frac{1}{2!}(-ig)\int d^4x\, \bar{\psi}_1\psi_1\phi_1(-ig)\int d^4y\, \bar{\psi}_2\psi_2\phi_2\,|\mathbf{p},\mathbf{k}\rangle \tag{5}$$

with the contractions

$$\overbrace{\psi_2\,|\mathbf{p}}\rangle\,, \quad \overbrace{\psi_1\,|\mathbf{k}}\rangle\,, \quad \langle\overbrace{\mathbf{k}'|\,\bar{\psi}_1}, \quad \langle\overbrace{\mathbf{p}'|\,\bar{\psi}_2}, \text{ and } \overbrace{\phi_1\phi_2}, \tag{6}$$

has the quantity as

$$(-ig)^2\int\frac{d^4q}{(2\pi)^4}\frac{i}{q^2-m_\phi^2}(2\phi)^2\delta(p'-p-q)\times(2\pi)^4\delta(k'-k+q)\bar{u}(p')u(p)\bar{u}(k')u(k). \tag{7}$$

c) Draw the corresponding Feynman diagram and prove the amplitude $i\mathcal{M}$ is given by

$$i\mathcal{M} = \frac{-ig^2}{q^2-m_\phi^2}\bar{u}(p')u(p)\bar{u}(k')u(k) \tag{8}$$

Exercise 2: Cross section for $e^+e^- \to \mu^+\mu^-$ (Written, 3+1 points)

In this exercise, we will calculate the unpolarized scattering cross section of the elementary QED process $e^+e^- \to \mu^+\mu^-$ to lowest order.

a) Draw the lowest order Feynman diagram for this process and calculate the scattering amplitude $i\mathcal{M}(e^-(p)e^+(p') \to \mu^-(k)\mu^+(k'))$.

b) Since we want to calculate the unpolarized scattering cross section, we have to average over all incoming spins and sum over the outgoing ones. Calculate

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 \tag{9}$$

and evaluate the traces.
Hint:

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{8e^4}{(p+p')^4}\left[(p'\cdot k)(p\cdot k') + (p'\cdot k')(p\cdot k) + p\cdot p' m_\mu^2 + k\cdot k' m_e^2 + 2m_\mu^2 m_e^2\right], \tag{10}$$

where m_μ and m_e are the masses of the muon and electron, respectively.

c) Specialize to the center-of-mass frame and calculate the differential cross section, $\mathrm{d}\sigma/\mathrm{d}\Omega$ and the total cross section σ in terms of the energy E of an incoming particle and the scattering angle θ.

d) **Bonus:** Calculate the differential and total cross section in the high-energy limit, $E \gg m_\mu, m_e$.