Prof. Dr. Hans Peter Büchler
Tutors: Jan Kumlin, Dr. Feiming Hu, and Dr. Daniel Huerga
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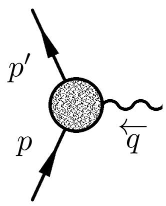
### Exercise 1: Rosenbluth formula (Oral)

As discussed in the lecture, the electromagnetic interaction vertex for a Dirac fermion can be written quite generally in terms of two form factors $F_1(q^2)$ and $F_2(q^2)$:




$$= \bar{u}(p') \left[ \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu} q_\nu}{2m} F_2(q^2) \right] u(p)$$

where $q = p' - p$ and $\sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu]$. If the fermion is a strongly interacting particle such as the proton, the form factors reflect the structure that results from the strong interactions and so are not easy to compute from first principles. However, these form factors can be determined experimentally. In this exercise, we will derive an expression for the cross section for the elastic scattering of an electron from a proton initially at rest to leading order in $\alpha$ but to all orders in the strong interactions.

a) Begin by first drawing the Feynman diagram of the scattering process

$$\text{proton}(p) + \text{electron}(k) \longrightarrow \text{proton}(p') + \text{electron}(k') \tag{1}$$

and write down the scattering amplitude for the process using the above rule for the vertex of the proton.

b) Compute the spin averaged and spin summed amplitude squared and evaluate all traces.
*Hint:* Use the *Gordon identity*

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p') \left[ \frac{(p'+p)^\mu}{2m} + \frac{i\sigma^{\mu\nu} q_\nu}{2m} \right] u(p) \tag{2}$$

in order to simplify the evaluation of the traces. The final result can be brought into the following form:

$$\begin{aligned} \frac{1}{4} \sum |\mathcal{M}|^2 \quad = \quad & \frac{8e^4}{q^4} \Big[ A((k' \cdot p)(k \cdot p) + (k' \cdot p)(k \cdot p') - (k \cdot k')m^2 - (p' \cdot p)m_e^2 + 2m_e^2 m^2) \\ & + B \left( k' \cdot (p'+p) k \cdot (p'+p) - \frac{(p'+p)^2}{2} (k' \cdot k - m_e^2) \right) \Big] , \end{aligned} \tag{3}$$

where $A$ and $B$ have to be determined.

c) Go into the lab frame, where the proton is at rest initially. Consider the electron energy $E \gg m_e$ (let $m_e \to 0$) and calculate the so called *Rosenbluth formula*

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta} = \frac{\pi\alpha\left[(F_1^2 - \frac{q^2}{4m^2}F_2^2)\cos^2\frac{\theta}{2} - \frac{q^2}{2m^2}(F_1+F_2)^2\sin^2\frac{\theta}{2}\right]}{2E^2\left[1+\frac{2E}{m}\sin^2\frac{\theta}{2}\right]\sin^4\frac{\theta}{2}}, \tag{4}$$

where $\theta$ is the lab-frame scattering angle and $F_1$ and $F_2$ are to be evaluated at the $q^2$ associated with elastic scattering at this angle.

By measuring $\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta}$ as a function of angle, it is thus possible to extract $F_1$ and $F_2$ experimentally.

### Exercise 2: Feynman Parameters (Written, 6 points)

We begin with the simpler case of two factors in the denominator,

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[xA+(1-x)B]^2} = \int_0^1 dxdy\delta(x+y-1)\frac{1}{[xA+yB]^2} \tag{5}$$

a) By differentiating with respect to $B$, prove that

$$\frac{1}{AB^n} = \int_0^1 dxdy\delta(x+y-1)\frac{ny^{n-1}}{[xA+yB]^{n+1}}. \tag{6}$$

b) By induction, prove

$$\frac{1}{A_1A_2\cdots A_n} = \int_0^1 dx_1dx_2\cdots dx_n\delta(\sum x_i - 1)\frac{(n-1)!}{[x_1A_1+x_2A_2+\cdots x_nA_n]^n}. \tag{7}$$

Hint: Use the conclusion from the previous task and note that

$$\begin{aligned} &\int_0^1 dx_1\int_0^1 dx_2\cdots\int_0^1 dx_n\delta(\sum x_i - 1)f(x_1,x_2,\cdots,x_n) \\ =&\int_0^1 dx_1\int_0^1 dx_2\cdots\int_0^{1-x_{n-2}} dx_{n-1}f(x_1,x_2,\cdots,x_{n-1},1-x_1-\cdots-x_{n-1}) \end{aligned} \tag{8}$$

c) By repeated differentiation of the above conclusion, prove

$$\frac{1}{A_1^{m_1}A_2^{m_2}\cdots A_n^{m_n}} = \int_0^1 dx_1dx_2\cdots dx_n\delta(\sum x_i - 1)\frac{\prod x_i^{m_i-1}}{[\sum x_iA_i]^{\sum m_i}}\frac{\Gamma(m_1+\cdots+m_n)}{\Gamma(m_1)\cdots\Gamma(m_n)}, \tag{9}$$

here $\Gamma$-function has $\Gamma(n)=(n-1)!$ for positive integers $n$.