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Exercise 1: Thomas-Fermi Screening (Oral)

In this exercise, we will apply the techniques from quantum field theory to condensed matter theory and study the Thomas-Fermi screening of the electron in a degenerate electron gas of density n at zero temperature.

a) Similar to the lecture, define $\Pi(q)$ to be the sum of all *one-particle-irreducible* diagrams contributing to the photon self-energy. Show by diagrammatically expanding the *full* photon propagator $D_{\text{ph}}(q)$ that

$$D_{\text{ph}}(q) = \frac{D^0_{\text{ph}}(q)}{1 - D^0_{\text{ph}}(q)\Pi(q)}, \tag{1}$$

where $D^0_{\text{ph}}(q)$ is the bare photon propagator. This is related to the so called *Lindhard theory* in condensed matter theory used for calculating the effects of electric field screening by electrons.

b) In condensed matter theory, the bare photon propagator in momentum space is simply the Fourier transform $V(q) \equiv U(\mathbf{q})$ of the (time-independent) interaction potential. Then, the denominator in (1) can be seen as a dielectric function given by (in the static limit)

$$\varepsilon(\mathbf{q}) = 1 - U(\mathbf{q})\Pi(\mathbf{q}). \tag{2}$$

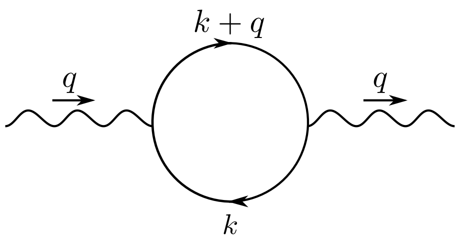
Show that the bare Coulomb interaction in momentum space is now modified to an effective interaction due to the screening of the electron gas in the long wavelength limit (i.e. Π is evaluated at $\mathbf{q} = 0$):

$$U_{\text{eff}}(\mathbf{q}) = \frac{e^2}{q^2 + \lambda_{\text{TF}}^{-2}}, \tag{3}$$

where λ_{TF}^{-1} is the Thomas-Fermi wave vector.

c) Calculate the Fourier transform of (3) and discuss your result.

d) **Bonus:** Calculate the Thomas-Fermi wave vector in the long wavelength limit ($\mathbf{q} \to 0$) and in the so-called *random-phase approximation*, where $\Pi(q)$ consists only of the particle-hole loop (neglecting the in- and outgoing lines)



According to the Feynman rules in condensed matter theory, $\Pi(q)$ is given by

$$\Pi(q) = -2i \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \int \frac{\mathrm{d}\omega}{2\pi} G^0(\omega, \mathbf{k}) G^0(\omega, \mathbf{k} + \mathbf{q}), \tag{4}$$

where

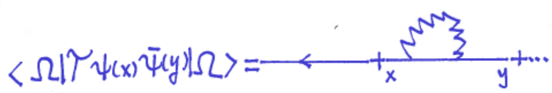
$$G^0(\omega, \mathbf{k}) = \frac{1}{\omega - \xi(\mathbf{k}) + i\delta\, \mathrm{sgn}(\xi(\mathbf{k}))} \tag{5}$$

with $\xi(\mathbf{k}) = \frac{\mathbf{k}^2}{2m} - E_F$ and E_F the Fermi energy. δ is to be taken positive but small (i.e. $\delta \to 0^+$) and $\mathrm{sgn}(x)$ refers to the signum function, which gives the sign of x and $\mathrm{sgn}(0) = 0$.
Hint:

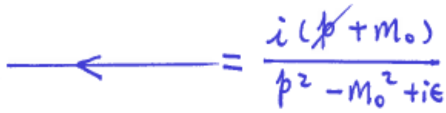
- In 3D, the Fermi energy is given by $E_F = (3\pi^2 n)^{2/3}/(2m)$.

Exercise 2: The electron self-energy (Written, 4 points)

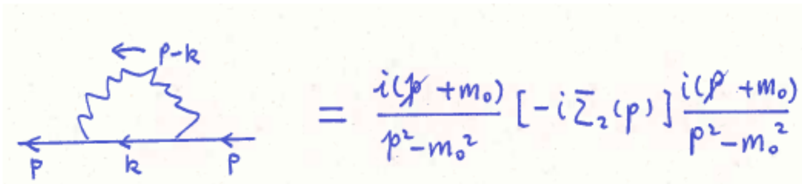
The electron two-point function is equal to the sum of diagrams:



The first diagram is just the free-field propagator:



and the second diagram, called the the *electron self-energy*, is somewhat more complicated:



where

$$-i\Sigma_2(p) = (-ie)^2 \int \frac{d^4 k}{(2\pi)^4} \gamma^\mu \frac{i(\not{k} + m_0)}{k^2 - m_0^2 + i\epsilon} \gamma_\mu \frac{-i}{(p-k)^2 - \mu^2 + i\epsilon}. \tag{6}$$

a) Using Feynman parameters, calculate the second-order self-energy $-i\Sigma_2(p)$.

b) By using the self-energy obtained in the last question, calculate and discuss the mass shift given by

$$\delta m = m - m_0 = \Sigma_2(\not{p} = m) \approx \Sigma_2(\not{p} = 0). \tag{7}$$

Is the mass shift ultraviolet divergent?