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Exercise 1: Dimensional regularization (Oral)

In this exercise we will work on the technical details of the *dimensional regularization*, due to 't Hooft and Veltman. Dimensional regularization preserves the symmetries of QED and also of a wide class of more general theories. Often, several different regulators can be used to renormalize a quantum field theory. Although the validity of our particular choice cannot be proven, we will use the one preserving symmetries in an axiomatic way.

The idea of dimensional regularization is based on writing the integral of a Feynman diagram as a function of dimensionality and taking the limit $d \to 4$ afterwards. Let us consider spacetime to have one time dimension and $(d-1)$ space dimensions. We may be interested in solving integrals of the form

$$\int \frac{d^d \ell_E}{(2\pi)^d} \frac{1}{(\ell_E^2 + \Delta)} = \int \frac{\Omega_d}{(2\pi)^d} \cdot \int d\ell_E \frac{\ell^{d-1}}{(\ell_E^2 + \Delta)^2} \tag{1}$$

where we have Wick rotated the time dimension.

a) The first factor in Eq. (1) contains the area of a unit sphere in d dimensions. Show that

$$\int d\Omega_d = \frac{2(\pi)^{d/2}}{\Gamma(d/2)} \tag{2}$$

Use $\sqrt{\pi} = \int dx e^{-x^2}$ and the definition of the Γ function.

b) Show that the second factor in Eq. (1) can be written as

$$\int \frac{d^d \ell_E}{(2\pi)^d} \frac{1}{(\ell_E^2 + \Delta)^2} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(2 - \frac{d}{2})}{\Gamma(2)} \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}. \tag{3}$$

Use the substitution $x = \Delta/(\ell^2 + \Delta)$ and the definition of the beta function,

$$\int_0^1 dx x^{\alpha-1}(1-x)^{\beta-1} = B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{4}$$

Where are the poles of the integral in Eq. (1)?

c) Define $\epsilon = 4 - d$ and use the infinite product representation $\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}$ –where γ is the Euler-Mascheroni constant– to expand $\Gamma(2 - \frac{d}{2})$ to first order in ϵ,

$$\Gamma\left(2 - \frac{d}{2}\right) = \Gamma(\epsilon/2) = \frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon). \tag{5}$$

d) Show that the integral gets the form

$$\int \frac{d^d\ell_E}{(2\pi)^d}\frac{1}{(\ell_E^2+\Delta)^2} \to \frac{1}{(4\pi)^2}\left(\frac{2}{\epsilon}-\gamma+\log\frac{4\pi}{\Delta}+\mathcal{O}(\epsilon)\right) \tag{6}$$

when $d \to 4$.

e) Following the previous steps verify that

$$\int \frac{d^d\ell_E}{(2\pi)^d}\frac{1}{(\ell_E^2+\Delta)^n} = \frac{1}{(4\pi)^{d/2}}\frac{\Gamma\left(n-\frac{d}{2}\right)}{\Gamma(n)}\left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}}, \tag{7}$$

$$\int \frac{d^d\ell_E}{(2\pi)^d}\frac{\ell_E^2}{(\ell_E^2+\Delta)^n} = \frac{1}{(4\pi)^{d/2}}\frac{d}{2}\frac{\Gamma\left(n-\frac{d}{2}-1\right)}{\Gamma(n)}\left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-1}. \tag{8}$$

These integrals will be used in the renormalization of the electric charge (next exercise).

Exercise 2: Computation of Π_2 (Written, 2 points)

In this exercise, we will apply the dimensional regularization to calculate $\Pi_2(q^2)$. From the lecture, you know

$$\begin{aligned} i\Pi_2^{\mu\nu}(q) &= -(-ie)^2\int\frac{d^4k}{(2\pi)^4}\mathrm{tr}\left[\gamma^\mu\frac{i(\not k+m)}{k^2-m^2}\gamma^\nu\frac{i(\not k+\not q+m)}{(k+q)^2-m^2}\right] \\ &= -4e^2\int\frac{d^4k}{(2\pi)^4}\frac{k^\mu(k+q)^\nu+k^\nu(k+q)^\mu-g^{\mu\nu}(k\cdot(k+q)-m^2)}{(k^2-m^2)((k+q)^2-m^2)}. \end{aligned} \tag{9}$$

After introducing Feynman parameter and new variable $l = k + xq$, one can perform a Wick rotation ($l^0 = il_E^0$) to obtain

$$\begin{aligned} i\Pi_2^{\mu\nu}(q) &= -4ie^2\int_0^1 dx\int\frac{d^4l_E}{(2\pi)^4} \\ &\times \frac{-\frac{1}{2}g^{\mu\nu}l_E^2+g^{\mu\nu}l_E^2-2x(1-x)q^\mu q^\nu+g^{\mu\nu}(m^2+x(1-x)q^2)}{(l_E^2+\Delta)^2}, \end{aligned} \tag{10}$$

where $\Delta = m^2 - x(1-x)q^2$.

a) By dimensional regularization, show that $i\Pi_2^{\mu\nu}(q)$ has the following form:

$$i\Pi_2^{\mu\nu}(q) = (q^2g^{\mu\nu}-q^\mu q^\nu)\cdot i\Pi_2(q^2), \tag{11}$$

where

$$\Pi_2(q^2) = \frac{-8e^2}{(4\pi)^{d/2}}\int_0^1 dx\, x(1-x)\frac{\Gamma\left(2-\frac{d}{2}\right)}{\Delta^{2-d/2}}. \tag{12}$$

Introduce $\epsilon = 4 - d$ and calculate the limit $\epsilon \to 0$.
Hint: In d dimensions, you have to replace the term $\frac{1}{2}g^{\mu\nu}l_E^2$ by $\frac{2}{d}g^{\mu\nu}l_E^2$.

b) Now we can compute the leading order-α shift in the electric charge (in the limit $\epsilon \to 0$):

$$e^2 - e_0^2 = \delta Z_3 = \Pi_2(0) \tag{13}$$

Further, calculate

$$\hat{\Pi}_2(q^2) \equiv \Pi_2(q^2) - \Pi_2(0), \tag{14}$$

and show that this is independent of ϵ.