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Exercise 1: Non-Abelian gauge theories and the Yang-Mills Lagrangian (Oral)

In this exercise, we will introduce the concept of *non-Abelian gauge invariance* in order to construct a broader class of field theories. As an example, we will discuss the *Yang-Mills* Lagrangian.

From the lecture, you know that the QED Lagrangian is invariant under a *local* U(1) transformation of the fermion field,

$$\psi(x) \to e^{i\alpha(x)}\psi(x), \tag{1}$$

This invariance is called *gauge symmetry*, and is a fundamental principle that determines the specific for of the Lagrangian. One can now generalize this principle of gauge invariance to other continuous symmetry groups. In this exercise, we will consider the SU(2) group, which consists of unitary 2×2 matrices with unit determinant.

Let us start by considering a doublet of Dirac fields

$$\psi = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}, \tag{2}$$

which transform under abstract three-dimensional rotations as a two-component spinor

$$\psi \to \exp\left(i\alpha^i \frac{\sigma^i}{2}\right)\psi, \tag{3}$$

where σ^i with $i = x, y, z$ are the Pauli matrices, i.e. the generators of the SU(2) group. We may now want to promote (3) to be a *local* symmetry of our theory by letting α^i be an arbitrary function of x,

$$\psi(x) \to V(x)\psi(x), \qquad V(x) = \exp\left(i\alpha^i(x)\frac{\sigma^i}{2}\right). \tag{4}$$

Since the generators σ^i of the symmetry group do not commute, we call this a *non-Abelian* symmetry group and the associated field theory will be called *non-Abelian gauge theory*. The *covariant derivative* associated with the local SU(2) symmetry is given by

$$D_\mu = \partial_\mu - igA^i_\mu \frac{\sigma^i}{2}, \tag{5}$$

where g is just a scaling factor and A^i are vector fields, also called *connections*. The second part of the the covariant derivative transforms as

$$A^i_\mu(x)\frac{\sigma^i}{2} \to V(x)\left(A^i_\mu(x)\frac{\sigma^i}{2} + \frac{i}{g}\partial_\mu\right)V^\dagger(x). \tag{6}$$

a) Show that for infinitesimal transformations, one has to first order in α

$$A_\mu^i \frac{\sigma^i}{2} \to A_\mu^i \frac{\sigma^i}{2} + \frac{1}{g}\left(\partial_\mu \alpha^i\right)\frac{\sigma^i}{2} + i\left[\alpha^i \frac{\sigma^i}{2}, A_\mu^j \frac{\sigma^i}{2}\right] + \cdots . \tag{7}$$

Show that, to first order in α, the covariant derivative transforms as

$$D_\mu \psi \to \left(1 + i\alpha^i \frac{\sigma^i}{2}\right) D_\mu \psi. \tag{8}$$

This can be also generalized to finite transformations $V(x)$. Thus, show that the term $\bar{\psi} i \not{D} \psi$ is gauge invariant under local transformations.

b) Show that, using the commutation relations of the Pauli matrices,

$$\left[\frac{\sigma^i}{2}, \frac{\sigma^i}{2}\right] = i\epsilon^{ijk}\frac{\sigma^k}{2}, \tag{9}$$

one has

$$[D_\mu, D_\nu] = -igF_{\mu\nu}^i \frac{\sigma^i}{2}, \tag{10}$$

with

$$F_{\mu\nu}^i = \partial_\mu A_\nu^i - \partial_\nu A_\mu^i + g\epsilon^{ijk} A_\mu^j A_\nu^k, \tag{11}$$

that is the generalized field-strength tensor.

c) Using

$$[D_\mu, D_\nu]\,\psi(x) \to V(x)\,[D_\mu, D_\nu]\,\psi(x), \tag{12}$$

write down the transformation for the object $F_{\mu\nu}^i \frac{\sigma^i}{2}$. Then, show that

$$\mathcal{L} = -\frac{1}{2}\mathrm{tr}\left[\left(F_{\mu\nu}^i\right)^2\right] = -\frac{1}{4}\left(F_{\mu\nu}^i\right)^2 \tag{13}$$

is gauge invariant. This Lagrangian describes the *Yang-Mills theory* and is the simplest example of a non-Abelian gauge symmetry. What is the difference of (13) compared to the Lagrangian of the electromagnetic field?

Combining (13) with the familiar Dirac Lagrangian, one arrives at the Yang-Mills Lagrangian

$$\mathcal{L} = \bar{\psi} i \not{D} \psi - \frac{1}{4}\left(F_{\mu\nu}^i\right)^2 - m\bar{\psi}\psi. \tag{14}$$