Exercise 1: Landau functional for the classical Heisenberg-model (10 points)

The Hamilton function for the classical Heisenberg-model is as follows:

$$H = -\frac{1}{2}\sum_{i,j} J_{ij}\boldsymbol{S}_i \cdot \boldsymbol{S}_j - \sum_i \boldsymbol{h}_i \cdot \boldsymbol{S}_i \ , \tag{1}$$

where $\boldsymbol{S}_i = \left(S_i^{(1)}, S_i^{(2)}, \ldots, S_i^{(N)}\right)$ is a classical vector with length $\mid \boldsymbol{S}_i \mid = 1 \ \forall i$. For J_{ij} we consider a general form satisfying $J_{ij} = J_{ji}$ and such that its Fourier-transform exists. The physical relevant cases are $N = 2$ (XY-model) and $N = 3$ (Heisenberg model). However, it is useful to consider N as a parameter, since an exact solution can be obtained in the limit $N = \infty$ (spherical model) and a controlled expansion can be performed around this solution ($1/N$-expansion).

Find the Landau functional for the model above with the help of the Hubbard-Stratonovich transformation

$$\exp\left\{\frac{\beta}{2}\sum_{i,j}\sum_{a=1}^{N} S_i^a J_{ij} S_j^a\right\} =$$
$$\frac{1}{(2\pi)^{MN/2}\left(\sqrt{\det J}\right)^N}\int \prod_{i=1}^{M} d\boldsymbol{\phi}_i \exp\left\{-\frac{1}{2\beta}\sum_{i,j}\sum_{a=1}^{N}\phi_i^a \left(J^{-1}\right)_{ij}\phi_j^a + \sum_{i=1}^{M}\sum_{a=1}^{N}\phi_i^a S_i^a\right\} \ , \tag{2}$$

where M is the number of lattice sites and $a = 1, \ldots, N$ denotes the components of $\boldsymbol{S}_i$ and $\boldsymbol{\phi}_i$.

Hints:

i) Take into account the fact that the variables $\boldsymbol{S}_i$ are subjected to a constraint.

$$\sum_{\{\boldsymbol{S}_i\}} \longrightarrow \int_{-\infty}^{\infty}\prod_i d^N S_i \ \delta(\boldsymbol{S}_i^2 - 1) \tag{3}$$

ii) In order to fulfill the constraint above, it is better to go over to spherical coordinates.

$$\begin{aligned}
S^{(1)} &= \cos\phi \sin\theta_1 \ldots \sin\theta_{N-2} \\
S^{(2)} &= \sin\phi \sin\theta_1 \ldots \sin\theta_{N-2} \\
S^{(3)} &= \cos\theta_1 \ldots \sin\theta_{N-2} \\
&\vdots \\
S^{(N-1)} &= \cos\theta_{N-3} \sin\theta_{N-2} \\
S^{(N)} &= \cos\theta_{N-2}
\end{aligned} \tag{4}$$

with $0 \leq \phi < 2\pi$ and $0 \leq \theta_i \leq \pi$ for $i = 1, \ldots, N-2$. The volume element goes over to an area element

$$d^N S \to d\phi \sin\theta_1 d\theta_1 \sin^2\theta_2 d\theta_2 \ldots \sin^{N-2}\theta_{N-2} d\theta_{N-2}$$

iii) The following integrals are usefull:

a)

$$\int_0^\pi \sin^k \theta d\theta = \frac{\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{k+2}{2}\right)} \tag{5}$$

Accordingly, the area of an N-dimensional sphere of radius unity is

$$S_N = \frac{2\pi^{N/2}}{\Gamma\left(\frac{N}{2}\right)}$$

b)

$$\int_0^\pi \sin^{N-2}\theta \, e^{B\cos\theta} d\theta = \sqrt{\pi}\left(\frac{2}{B}\right)^{\frac{N}{2}-1} \Gamma\left(\frac{N-1}{2}\right) I_{\frac{N}{2}-1}(B) \,, \tag{6}$$

where $I_\nu(z)$ is a modified Bessel-function, with

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^\infty \frac{(z/2)^{2k}}{k!\Gamma(\nu+k+1)} \,, \tag{7}$$

where $\nu \neq -1, -2, \ldots$.

c) Using a) and b) one has

$$\int \mathcal{D}\boldsymbol{S}\, \delta(\boldsymbol{S}^2 - 1) \exp\left\{\beta \sum_i^M \boldsymbol{\phi}_i \cdot \boldsymbol{S}_i\right\} = \prod_{i=1}^M \left\{2\pi^{\frac{N}{2}} \left(\frac{2}{\beta \mid \boldsymbol{\phi}_i \mid}\right)^{\frac{N}{2}-1} I_{\frac{N}{2}-1}(\beta \mid \boldsymbol{\phi}_i \mid)\right\} \tag{8}$$

Exercise 2: ϕ^3-theory (10 points)

The Action for the ϕ^3-theory in d-dimensions is given by

$$S = \int \mathrm{d}^d r \left[\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2} m_0^2 \phi^2 + \frac{\lambda}{3!}\phi^3\right] \tag{9}$$

i) Find the upper critical dimension of the theory.

ii) Find the Hamiltonian density corresponding to the given ϕ^3-action.

iii) Can the action in (9) describe a real physical system? why?

iv) For the case of $\lambda = 0$ (non-interacting theory), compute explicitly the 6-point correlation function $\langle \phi(\mathbf{x}_1)\phi(\mathbf{x}_2)\phi(\mathbf{x}_3)\phi(\mathbf{x}_4)\phi(\mathbf{x}_5)\phi(\mathbf{x}_6)\rangle$.