Spin waves in the Heisenberg model

The Hamiltonian for the Heisenberg model is given by

$$H = J \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j, \tag{1}$$

where the spin-variables $\mathbf{S}_i$ satisfy the SU(2) commutation relation

$$[S_i^a, S_j^b] = i\delta_{ij}\epsilon_{abc}S_i^c, \qquad a, b, c = x, y, z, \tag{2}$$

where ϵ is the antisymmetric tensor. For simplicity, we restrict ourselves to nearest-neighbour interaction. This means that the sum in (1) runs over pairs of nearest neighbours. We consider a d-dimensional cubic lattice.

<u>Exercise 6:</u> Heisenberg ferromagnet (10 points)

In the ferromagnetic case $J < 0$. We define the operators S^+ and S^- as

$$S^{\pm} = S^x \pm iS^y. \tag{3}$$

i) Show that the operators S^+, S^- and S^z satisfy the commutation relations

$$[S^z, S^+] = S^+, \quad [S^z, S^-] = -S^-, \quad [S^+, S^-] = 2S^z. \tag{4}$$

ii) Write the Hamiltonian (1) in terms of the operators S^+, S^- and S^z and show that the state of all spin pointing in the same direction $|0\rangle = |\uparrow\uparrow\uparrow \ldots\rangle$ is the exact ground state of the system.

iii) Show that the state $|\mathbf{k}\rangle$ defined as

$$|\mathbf{k}\rangle = \sum_i e^{i\mathbf{k}\cdot\mathbf{R}_i} S_i^- |0\rangle \tag{5}$$

is an exact eigenstate of the system and find its energy relative to the ground state as a function of $\mathbf{k}$ (Hint: consider the commutator $[H, S_i^-]$).

<u>Exercise 7:</u> Heisenberg antiferromagnet and the Holstein-Primakoff transformation (10 points)

In the antiferromagnetic case $J > 0$. It is convenient in this case to divide the lattice into sublattices A and B such that nearest neighbours always belong to different sublattices. We can redefine the spin operators by rotating the spin in sublattice B such that

$$\tilde{S}_{A,B}^z = \pm S^z, \quad \tilde{S}_{A,B}^+ = S^x \pm iS^y, \quad \tilde{S}_{A,B}^- = S^x \mp iS^y. \tag{6}$$

i) The Néel state is the state where all the spins point on one direction in one sublattice and in the opposite direction in the other $|N\rangle = |\uparrow\uparrow\uparrow\ldots\rangle_A \otimes |\downarrow\downarrow\downarrow\ldots\rangle_B$. It is the ground state for the classical antiferromagnetic Heisenberg model. Show that it cannot be the ground state in the quantum case.

ii) Although the Néel state is not the actual ground state, in the semiclassical limit $S \gg 1$, it is a very good approximation to the ground state and we will consider fluctuation on top of it. In order to do this, we define a_i as

$$\tilde{S}_i^- = a_i^\dagger \left(2S - a_i^\dagger a_i\right)^{1/2}, \quad \tilde{S}_i^+ = \left(2S - a_i^\dagger a_i\right)^{1/2} a_i, \quad \tilde{S}_i^z = S - a_i^\dagger a_i. \tag{7}$$

Show that if the variable a satisfy bosonic canonical commutation relations, the variables $\tilde{S}$ will satisfy the spin SU(2) commutation relations.

iii) Write the Hamiltonian (1) in terms of the bosonic variables a_i and expand in powers of $1/S$ keeping the two lowest order terms.

iv) Show that the resulting quadratic Hamiltonian can be diagonalized and find the dispersion relation for the resulting modes.