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May 17th, 2017
SS 2017

Exercise 1: Correlation functions in a Fermi sea (Oral)

Consider a gas of N identical fermions with spin $\frac{1}{2}$. The fermions are free and non-interacting and the ground state of the system is given by the Fermi sea

$$|\mathrm{FS}\rangle = \prod_{|\boldsymbol{k}|<k_F,\sigma} c^\dagger_{\boldsymbol{k},\sigma}|0\rangle \,. \tag{1}$$

One defines the one-particle correlation function $G_\sigma(\boldsymbol{r}-\boldsymbol{r}')$ as

$$G_\sigma(\boldsymbol{r}-\boldsymbol{r}') = \frac{n}{2} g_\sigma(\boldsymbol{r}-\boldsymbol{r}') = \langle \mathrm{FS}|\Psi^\dagger_\sigma(\boldsymbol{r})\Psi_\sigma(\boldsymbol{r}')|\mathrm{FS}\rangle \,. \tag{2}$$

This is the amplitude of creating a fermion of spin σ at position $\boldsymbol{r}$ when one was annihilated at position $\boldsymbol{r}'$ with the same spin.

a) Express the field operators in the natural basis, that is

$$\Psi_\sigma(\boldsymbol{r}) = \frac{1}{\sqrt{V}} \sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\boldsymbol{r}} c_{\boldsymbol{k},\sigma} \,, \tag{3}$$

where $c^\dagger_{\boldsymbol{k},\sigma}$ and $c_{\boldsymbol{k},\sigma}$ are the creation and annihilation operators, respectively. Calculate $G_\sigma(\boldsymbol{r}-\boldsymbol{r}')$ and sketch its graph as a function of $k_F|\boldsymbol{r}-\boldsymbol{r}'|$.

b) Discuss its behaviour both for $k_F|\boldsymbol{r}-\boldsymbol{r}'| \to \infty$ and $k_F|\boldsymbol{r}-\boldsymbol{r}'| \to 0$.

c) Likewise, one can define the pair correlation function $g_{\sigma\sigma'}(\boldsymbol{r}-\boldsymbol{r}')$ by

$$\left(\frac{n}{2}\right)^2 g_{\sigma\sigma'}(\boldsymbol{r}-\boldsymbol{r}') = \langle \mathrm{FS}|\Psi^\dagger_\sigma(\boldsymbol{r})\Psi^\dagger_{\sigma'}(\boldsymbol{r}')\Psi_{\sigma'}(\boldsymbol{r}')\Psi_\sigma(\boldsymbol{r})|\mathrm{FS}\rangle \,. \tag{4}$$

Assume first that $\sigma \neq \sigma'$. Calculate $g_{\sigma\sigma'}(\boldsymbol{r}-\boldsymbol{r}')$.

d) Now consider the case where $\sigma = \sigma'$ and calculate $g_{\sigma\sigma}(\boldsymbol{r}-\boldsymbol{r}')$. Plot $g_{\sigma\sigma}(\boldsymbol{r}-\boldsymbol{r}')$ as a function of $k_F|\boldsymbol{r}-\boldsymbol{r}'|$.

e) Show that the size of the dip in the correlation function corresponds to the displacement of one electron.

Exercise 2: The Bogoliubov-Valatin transformation (Written)

The Hamiltonian (in the grand-canonical ensemble) for the free electron gas may be written in 2-component form as

$$K = \sum_{\boldsymbol{k}} \xi_{\boldsymbol{k}} : \left(c^\dagger_{\boldsymbol{k},\uparrow}, c_{-\boldsymbol{k},\downarrow}\right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} c_{\boldsymbol{k},\uparrow} \\ c^\dagger_{-\boldsymbol{k},\downarrow} \end{pmatrix} : \tag{5}$$

where $\xi_{\boldsymbol{k}} = \epsilon_{\boldsymbol{k}} - \mu$ with the chemical potential μ. The Bogoliubov-Valatin transformation

$$\begin{pmatrix} c_{\boldsymbol{k},\uparrow} \\ c^\dagger_{-\boldsymbol{k},\downarrow} \end{pmatrix} = U_{\boldsymbol{k}} \begin{pmatrix} \alpha_{\boldsymbol{k},\uparrow} \\ \alpha^\dagger_{-\boldsymbol{k},\downarrow} \end{pmatrix} \tag{6}$$

expresses spin-up excitations of a given momentum $\boldsymbol{k}$ in electron-hole space through new quasiparticle operators α ($U_{\boldsymbol{k}}$ is a 2×2 matrix).

a) Show that if and only if the transformation is unitary the (fermionic) commutation relations are preserved.

b) Choosing $U_{\boldsymbol{k}}$ in $SU(2)$ (so that $\det U_{\boldsymbol{k}} = 1$), we have

$$U_{\boldsymbol{k}} = \begin{pmatrix} u_{\boldsymbol{k}} & -v^*_{\boldsymbol{k}} \\ v_{\boldsymbol{k}} & u^*_{\boldsymbol{k}} \end{pmatrix} . \tag{7}$$

In order to describe states below the Fermi surface in the hole language and states above the Fermi surface in the electron language, we choose $u_{\boldsymbol{k}} = 1$, $v_{\boldsymbol{k}} = 0$ for $k > k_F$ and $u_{\boldsymbol{k}} = 0$, $v_{\boldsymbol{k}} = 1$ for $k < k_F$. Show that the Hamiltonian then takes the form

$$K = \sum_{\boldsymbol{k}} |\xi_{\boldsymbol{k}}| \left(\alpha^\dagger_{\boldsymbol{k},\uparrow} \alpha_{\boldsymbol{k},\uparrow} + \alpha^\dagger_{\boldsymbol{k},\downarrow} \alpha_{\boldsymbol{k},\downarrow} \right) + \sum_{k<k_F} \xi_{\boldsymbol{k}} . \tag{8}$$

Note: In the above exercise, the matrix elements $u_{\boldsymbol{k}}$ and $v_{\boldsymbol{k}}$ satisfied $u_{\boldsymbol{k}} v_{\boldsymbol{k}} = 0$. Later, in the BCS-theory of superconductivity, the product of these matrix elements will not vanish, that is $u_{\boldsymbol{k}} v_{\boldsymbol{k}} \neq 0$. This means that there is a pairing mechanism between electrons with opposite spin and opposite momenta.