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Exercise 1: Current conservation (2 points, Oral)

Consider the continuity equation in real space,

$$\partial_t \rho(x) = -\frac{\mathrm{i}}{\hbar}\left[\rho(x), H\right], \tag{1}$$

and a general Hamiltonian of interacting fermions in second quantized form,

$$H = \int d^3x\, \frac{\hbar^2}{2m} \nabla\psi^\dagger(x)\nabla\psi(x) + \frac{1}{2}\int d^3x\, d^3y\, \psi^\dagger(x)\psi^\dagger(y)\psi(y)\psi(x)V(x-y), \tag{2}$$

to show that

$$\partial_t \rho(x) = -\nabla j(x), \tag{3}$$

where the current is

$$j(x) = -\frac{\mathrm{i}}{2m}\left[(\nabla\psi^\dagger(x))\psi(x) - \psi^\dagger(x)\nabla\psi(x)\right], \tag{4}$$

and the density operator in terms of fields was defined in the lecture, $\rho(x) = \psi^\dagger(x)\psi(x)$.

Exercise 2: Sum rules (5 points, Oral)

In this exercise, we are going to prove the sum rule of the structure factor, $S(\boldsymbol{q}, w)$,

$$\int_0^\infty dw\, w\, S(q, w) = \frac{N}{V}\frac{q^2}{2m}, \tag{5}$$

implying the following limit of the density-correlator,

$$\chi(q, w \to \infty) = \frac{N}{V}\frac{q^2}{2m}. \tag{6}$$

a) Fourier transform $j(x)$ to obtain the current in the momentum space,

$$j_q = \sum_{k,\sigma} [(k + q/2)/m] c^\dagger_{k,\sigma} c_{k+q,\sigma}. \tag{7}$$

b) Show that, in momentum space, the continuity equation (1) leads to,

$$[\rho_q, H] = -\mathrm{i}\hbar\, q \cdot j_q. \tag{8}$$

c) Show that

$$[[\rho_q, H], \rho_{-q}] = [q \cdot j_q, \rho_{-q}] = \frac{Nq^2}{m}. \tag{9}$$

d) Evaluate Eq. (9) in the ground-state, i.e. the Fermi sea $|\Psi_0\rangle$, to show,

$$\langle\Psi_0| [[\rho_q, H], \rho_{-q}] |\Psi_0\rangle = \frac{2}{V} \int_0^\infty dw\, w\, S(q, w). \tag{10}$$

For this purpose, first use the first commutator equality of task b) and introduce the identity, $\sum_n |n\rangle \langle n|$,

$$\langle\Psi_0| [[\rho_q, H], \rho_{-q}] |\Psi_0\rangle = \sum_n \langle 0| q \cdot j_q |n\rangle \langle n| \rho_{-q} |0\rangle - \sum_n \langle 0| \rho_{-q} |n\rangle \langle n| q \cdot j_q |0\rangle . \tag{11}$$

Then show that $\langle n| q \cdot j_q |0\rangle = -w_{n0} \langle n| \rho_q |0\rangle$, and convert the sum into a continuous integral in w.

e) Use this result to show that

$$\chi(q, w \to \infty) = \frac{2}{w^2} \int_0^\infty dw'\, w'\, S(q, w') = \frac{N}{V} \frac{q^2}{mw^2}. \tag{12}$$

Exercise 3: Response function of the free Fermi gas (2 points, Written)

Consider the response function seen in the course,

$$\chi(q, w) = \frac{1}{V\hbar} \sum_n |\langle n|\rho_{-q}|\Psi_0\rangle|^2 \left(\frac{1}{w - w_{n0} + \mathrm{i}\eta} - \frac{1}{w + w_{n0} + \mathrm{i}\eta} \right). \tag{13}$$

Show that, for the free Fermi gas, it can be written as

$$\chi(q, w) = 2 \int \frac{dk}{(2\pi)^3} \frac{n_k - n_{k+q}}{\hbar w - (\varepsilon_{k+q} - \varepsilon_k) + \mathrm{i}\eta}, \tag{14}$$

where the energies are $\varepsilon_k = \hbar^2 k^2 / 2m$, and n_k is the Fermi-Dirac distribution.

Hint: Consider the possible excitation states $|n\rangle$ of a free Fermi gas, i.e. particle-hole excitations.