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Exercise 1: Quasi-particles at finite temperatures (Oral)

The distribution function of a Fermi liquid in the thermodynamic equilibrium is given by

$$n_p = \frac{1}{1 + e^{(\tilde{\epsilon}_p - \mu)/T}}\,, \tag{1}$$

with the quasi-particle energy $\tilde{\epsilon}_p = \epsilon_p + \sum_{p'} f_{p,p'} \delta n_{p'}$ and $\delta n_p = n_p - n_p^0$ ($n_p^0 = \theta(p - p_F)$). Show that at low temperatures

$$\sum_{p'} f_{p,p'} \delta n_{p'} \propto T^2 V g(\mu) \frac{f}{m^* v_F^2}\,, \tag{2}$$

with $g(\mu) = m^* p_F / \pi^2$ and f as an energy scale for $f_{p,p'}$. Argue that at low temperatures, one can replace the quasi-particle energy $\tilde{\epsilon}_p$ by the non-interacting quasi-particle energy ϵ_p in (1).

Exercise 2: Quasi-particle current (Written, 4 points)

The transport equation for the distribution function in the absence of external field or collisions has the form

$$\frac{\partial n_p}{\partial t} + \nabla_r n_p \cdot \nabla_p \tilde{\epsilon}_p - \nabla_p n_p \cdot \nabla_r \tilde{\epsilon}_p = 0\,. \tag{3}$$

$\nabla_p \tilde{\epsilon}_p$ is the velocity of the quasi-particle and $-\nabla_r \tilde{\epsilon}_p$ the force it feels.

a) Show by inserting $n_p(r,t) = n_p^0 + \delta n_p(r,t)$ and expanding to first order in δn_p that this leads to a transport equation for the quasi-particles:

$$\frac{\partial \delta n_p}{\partial t} + \nabla_r \delta n_p \cdot v_p - \nabla_p n_p^0 \cdot \sum_{p'} f_{p,p'} \nabla_r \delta n_{p'} = 0\,. \tag{4}$$

b) Interpret (4) as a continuity equation for the current and density of the quasi-particles and derive the quasi-particle current

$$\boldsymbol{J} = \sum_p \delta n_p \boldsymbol{j}_p \tag{5}$$

with the current for a single quasi-particle with momentum $\boldsymbol{p}$

$$\boldsymbol{j}_p = \boldsymbol{v}_p - \sum_{p'} f_{p,p'} \frac{\partial n_p^0}{\partial \epsilon_{p'}} \boldsymbol{v}'_p\,. \tag{6}$$

The second term in (6) has the interpretation of a 'drag' or 'backflow' current which is created by the quasi-particle dragging the medium due to the interactions. The origin of this 'drag current' can be seen from the derivation of the quasi-particle current using Galilean invariance which in turn is only valid in translationally invariant systems. Upon applying a Galilean transformation the quasi-particle's momentum is changed by $\boldsymbol{q}$ ($\boldsymbol{p} \to \boldsymbol{p} + \boldsymbol{q}$) if $\boldsymbol{q}/m = \boldsymbol{v}$ is the velocity in the moving inertial frame of reference. The resulting change in the kinetic energy of the system is thus given in first order in $\boldsymbol{q}$ as

$$\delta E = \left\langle \sum_i \boldsymbol{q} \cdot \frac{\boldsymbol{p}_i}{m} \right\rangle = \langle \boldsymbol{q} \cdot \hat{\boldsymbol{J}} \rangle = \boldsymbol{q} \cdot \boldsymbol{J} \,, \tag{7}$$

where we defined $\hat{\boldsymbol{J}} = \sum_i \boldsymbol{p}_i/m$ as the total current operator. The total current then can be calculated from $\boldsymbol{J} = \nabla_q E$.

c) In the following, we apply this relation onto the state of an single excited quasi-particle with momentum $\boldsymbol{p}$ (so $\boldsymbol{J}$ reduces to $\boldsymbol{j}_p$). Upon a Galilean transformation, both the quasi-particle's momentum as well as the occupation of the Fermi sea change. Calculate the change in energy of the system and the resulting quasi-particle current $\boldsymbol{j}_p$.

d) For a system which is invariant under any translation, the total momentum $\boldsymbol{P}$ of the system is a good quantum number. The total current is then given by $\boldsymbol{J} = \boldsymbol{P}/m$. For a state containing only a single excited quasi-particle of momentum $\boldsymbol{p}$, the current is thus given by $\boldsymbol{j}_p = \boldsymbol{p}/m$. Using the quasi-particle current calculated in the previous task, show that the effective mass is given by

$$m^* = m \left(1 + \frac{F_1^s}{3}\right) \,. \tag{8}$$

The system is therefore stable for $F_1^s > -3$.

Exercise 3: Spin susceptibility (Oral)

a) Derive the paramagnetic spin susceptibility for a non-interacting electron gas at $T = 0$, given by

$$\chi^0 = g^0(\mu)\mu_B^2 = \frac{3n}{2\epsilon_F^0}\mu_B^2 \,, \tag{9}$$

with $g^0(\mu)$ the density of states and ϵ_F^0 the Fermi energy of the non-interacting system.

b) Show that within Fermi liquid theory the susceptibility at $T = 0$ is given by

$$\chi = \frac{g(\mu)}{1 + F_0^a}\mu_B^2 = \frac{1 + F_1^s/3}{1 + F_0^a}\chi^0 \,, \tag{10}$$

where $g(\mu)$ is the density of states of the interacting system.