Exercise 1: Quasi-particles at finite temperatures (Oral)

The distribution function of a Fermi liquid in the thermodynamic equilibrium is given by

$$n_p = \frac{1}{1 + e^{(\tilde{\epsilon}_p - \mu)/T}},$$

(1)

with the quasi-particle energy

$$\tilde{\epsilon}_p = \epsilon_p + \sum_{p'} f_{p,p'} \delta n_{p'}$$

and

$$\delta n_p = n_p - n_p^0 \quad (n_p^0 = \theta(p - p_F)).$$

Show that at low temperatures

$$\sum_{p'} f_{p,p'} \delta n_{p'} \propto T^2 V g(\mu) \frac{f}{m^* v_F},$$

(2)

with $g(\mu) = m^* p_F / \pi^2$ and $f$ as an energy scale for $f_{p,p'}$. Argue that at low temperatures, one can replace the quasi-particle energy $\tilde{\epsilon}_p$ by the non-interacting quasi-particle energy $\epsilon_p$ in (1).

Exercise 2: Quasi-particle current (Written, 4 points)

The transport equation for the distribution function in the absence of external field or collisions has the form

$$\frac{\partial n_p}{\partial t} + \nabla_r n_p \cdot \nabla \tilde{\epsilon}_p - \nabla_p n_p \cdot \nabla_r \tilde{\epsilon}_p = 0.$$

(3)

$\nabla_p \tilde{\epsilon}_p$ is the velocity of the quasi-particle and $-\nabla_r \tilde{\epsilon}_p$ the force it feels.

a) Show by inserting $n_p(r,t) = n_p^0 + \delta n_p(r,t)$ and expanding to first order in $\delta n_p$, that this leads to a transport equation for the quasi-particles:

$$\frac{\partial \delta n_p}{\partial t} + \nabla_r \delta n_p \cdot v_p - \nabla_p n_p^0 \cdot \sum_{p'} f_{p,p'} \nabla_r \delta n_{p'} = 0.$$

(4)

b) Interpret (4) as a continuity equation for the current and density of the quasi-particles and derive the quasi-particle current

$$J_p = \sum_p \delta n_p j_p$$

(5)

with the current for a single quasi-particle with momentum $p$

$$j_p = v_p - \sum_{p'} f_{p,p'} \frac{\partial n_{p'}^0}{\partial \epsilon_{p'}}.$$

(6)
The second term in (6) has the interpretation of a 'drag' or 'backflow' current which is created by the quasi-particle dragging the medium due to the interactions. The origin of this 'drag current' can be seen from the derivation of the quasi-particle current using Galilean invariance which in turn is only valid in translationally invariant systems. Upon applying a Galilean transformation the quasi-particle’s momentum is changed by $q$ ($p \rightarrow p + q$) if $q/m = v$ is the velocity in the moving inertial frame of reference. The resulting change in the kinetic energy of the system is thus given in first order in $q$ as

$$\delta E = \langle \sum_i q \cdot \frac{p_i}{m} \rangle = \langle q \cdot \hat{J} \rangle = q \cdot J,$$

(7)

where we defined $\hat{J} = \sum_i p_i/m$ as the total current operator. The total current then can be calculated from $J = \nabla_q E$.

c) In the following, we apply this relation onto the state of a single excited quasi-particle with momentum $p$ (so $J$ reduces to $j_p$). Upon a Galilean transformation, both the quasi-particle’s momentum as well as the occupation of the Fermi sea change. Calculate the change in energy of the system and the resulting quasi-particle current $j_p$.

d) For a system which is invariant under any translation, the total momentum $P$ of the system is a good quantum number. The total current is then given by $J = P/m$. For a state containing only a single excited quasi-particle of momentum $p$, the current is thus given by $j_p = p/m$. Using the quasi-particle current calculated in the previous task, show that the effective mass is given by

$$m^* = m \left(1 + \frac{F_s}{3}\right).$$

(8)

The system is therefore stable for $F^s_1 > -3$.

**Exercise 3: Spin susceptibility (Oral)**

a) Derive the paramagnetic spin susceptibility for a non-interacting electron gas at $T = 0$, given by

$$\chi^0 = g^0(\mu) \mu_B^2 = \frac{3n}{2\epsilon_F^0} \mu_B^2,$$

(9)

with $g^0(\mu)$ the density of states and $\epsilon_F^0$ the Fermi energy of the non-interacting system.

b) Show that within Fermi liquid theory the susceptibility at $T = 0$ is given by

$$\chi = \frac{g(\mu)}{1 + \frac{\mu_B}{\mu_B}} = \frac{1 + F^s_1/3}{1 + \frac{\mu_B}{\mu_B}} \chi^0,$$

(10)

where $g(\mu)$ is the density of states of the interacting system.