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Exercise 1: Thomas–Fermi Screening (Oral)

The susceptibility $\chi_{\rm sc}(\boldsymbol{q})$ is defined in terms of the Fourier transforms of the charge density $\rho_{\rm ind}$ and the total potential $\phi(\boldsymbol{r})$ (incuding internal and external sources) by

$$\rho_{\rm ind}(\boldsymbol{q}) = \chi_{\rm sc}(\boldsymbol{q})\chi(\boldsymbol{q}). \tag{1}$$

In the Thomas-Fermi theory the induced charge density of the equilibrium distribution of particles in an external potential is taken to be

$$\rho_{\rm ind}(\boldsymbol{r}) = -e\{n_0\left[\mu + e\phi(\boldsymbol{r})\right] - n_0(\mu)\}, \tag{2}$$

where μ is the chemical potential in the absence of an electric potential.

a) Under the assumption that $\phi(\boldsymbol{r})$ is small, write $\rho_{\rm ind}(\boldsymbol{r})$ as a derivative and find from this an expression for $\chi_{\rm sc}(\boldsymbol{q})$. What is $\chi_{\rm sc}(\boldsymbol{q})$ for an electron gas?

b) The static dielectric function is defined through $\epsilon(\boldsymbol{q}) = 1 - (4\pi e/q^2)\chi_{\rm sc}(\boldsymbol{q})$ for all $\boldsymbol{q}$. Write $\epsilon(\boldsymbol{q})$ for the electron gas and from this find an expression for the total potential $\phi(\boldsymbol{q})$ in the presence of an extra Coulomb charge, $\phi_{\rm Coulomb}(\boldsymbol{q}) = 4\pi/q^2$. What is the Thomas–Fermi screening length $q_{\rm TF}^{-1}$ for the electron gas? Transform $\phi(\boldsymbol{q})$ back into real space and give a short interpretation of $q_{\rm TF}^{-1}$.

Exercise 2: Landau–Silin Theory (Oral)

The Landau–Silin transport equation in the presence of an electric field is

$$(-\omega + \boldsymbol{q}\cdot\boldsymbol{v_p})\delta n_{\boldsymbol{p}} + \left[\boldsymbol{q}\cdot\boldsymbol{v_p}\left(\sum_{\boldsymbol{p}'} f_{\boldsymbol{pp}'}\delta n_{\boldsymbol{p}'}\right) + ie\boldsymbol{E}\cdot\boldsymbol{v_p}\right]\delta(\epsilon_{\boldsymbol{p}} - \mu) = 0 \tag{3}$$

a) Using the simple model $f_{\boldsymbol{pp}'} = f_0 \equiv (\pi^2\hbar^2/m^* k_F)F_0$, find an expression for the density $\langle\rho(\boldsymbol{q},\omega)\rangle = \sum_{\boldsymbol{p}}\delta n_{\boldsymbol{p}}$ induced in response to a longitudinal field $\boldsymbol{E}(\boldsymbol{q},\omega) \parallel \boldsymbol{q}$. Your answer should be written in terms of the function

$$F(\lambda) = 1 - (\lambda/2)\log[(\lambda + i\eta + 1)/(\lambda + i\eta - 1)] \tag{4}$$

where $\lambda = \omega/v_F q$.

The longitudinal dielectric constant is

$$\epsilon(\boldsymbol{q},\omega) = 1 + \frac{4\pi ie\langle\rho(\boldsymbol{q},\omega)\rangle}{\boldsymbol{q}.\boldsymbol{E}(\boldsymbol{q},\omega)} \tag{5}$$

b) Use your answer for $\langle \rho(\boldsymbol{q}, \omega) \rangle$ to show that for the simple model,

$$\epsilon(\boldsymbol{q}, \omega) = 1 + \frac{3}{v_F^2 q^2} \frac{\omega_p^2 F(\lambda)}{1 + F_0 F(\lambda)}. \tag{6}$$

c) Discuss the form of Im ϵ and Re ϵ as a function of $\omega / q v_F$ for the different regimes of F_0. In the case of strong coupling ($F_0 \gg 1$) show that the zero sound pole in $\chi(\boldsymbol{q}, \omega) = (q^2/4\pi) [\epsilon(\boldsymbol{q}, \omega) - 1]$ occurs at large frequencies, $\lambda \simeq \sqrt{F_0/3}$. (Hint: expand $F(\lambda)$ in powers of $1/\lambda$.) Show that at large frequencies the dielectric constant may be written,

$$\epsilon(\boldsymbol{q}, \omega) = 1 - \frac{\omega_p^2}{(\omega + i\eta)^2 - s^2 q^2}, \tag{7}$$

where the sound velocity is $s = v_F \sqrt{F_0/3}$.

d) What will happen in the response to an external field when the frequency and wavelength of the probe satisfy $\omega = sq$?