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June 5th, 2017
SS 2017

Exercise 1: Canonical transformations (Oral, 2 points).

Show that the linear mapping $\{c_{k\sigma}, c_{k\sigma}^\dagger\} \to \{\alpha_{k\sigma}, \alpha_{k\sigma}^\dagger\}$ proposed by Bogoliubov–Valatin,

$$c_{k\uparrow} = u_k^* \alpha_{k\uparrow} + v_k \alpha_{-k\downarrow}^\dagger, \tag{1}$$

$$c_{-k\downarrow} = -v_k^* \alpha_{k\uparrow} + u_k \alpha_{-k\downarrow}^\dagger, \tag{2}$$

is canonical —i.e. it preserves the fermionic canonical commutation relations.

Exercise 2: Poor man's Bogoliubov–Valatin (Written, 2 points).

Plug the Bogoliubov–Valatin transformation (1)–(2) directly in the BCS Hamiltonian,

$$H = \sum_{k\sigma} \xi_k c_{k\sigma}^\dagger c_{k\sigma} - \sum_k \left(\Delta_k c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger + \Delta_k^* c_{-k\downarrow} c_{k\uparrow} - \Delta_k \langle c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger \rangle \right), \tag{3}$$

and require the resulting expression to be diagonal —i.e. proportional to the number operator of quasi-particles, $\alpha_{k\sigma}^\dagger \alpha_{k\sigma}$, plus a constant— to obtain,

$$2\xi_k u_k v_k + \Delta_k^* v_k^2 - \Delta_k u_k^2 = 0, \tag{4}$$

which gives you the BCS gap equation when multiplying by Δ_k^*/u_k^2.

Exercise 3: Long-range order (Oral, 2 points).

In this exercise we will study the behaviour of the pair correlator,

$$\langle \psi_\uparrow^\dagger(r) \psi_\downarrow^\dagger(r) \psi_\uparrow(0) \psi_\downarrow(0) \rangle, \tag{5}$$

which signals the onset/absence of superconducting *long-range order*. Compute it with the BCS ansatz at long-distances for $T = 0$,

a) out of the superconducting phase,

b) in the superconducting phase.

Recall the definition of the field operator,

$$\psi_\sigma(r) = \frac{1}{\sqrt{V}} \sum_k e^{\mathrm{i}kr} c_{k\sigma}. \tag{6}$$