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You can find detailed information about the lecture and the exercises on the website `http://www.itp3.uni-stuttgart.de/lehre/vorlesungen/QM-main.html`. There are two types of exercises: *Written* exercises have to be handed in and will be corrected by the tutors. *Oral* exercises have to be prepared for the exercise session and will be presented by one of the students. To be admitted to the exam, we require (a) 80% of the written exercises to be solved or treated adequately, (b) 66% of the oral exercises to be prepared and (c) two exercises to be presented on the black board.

Exercise 1: Plane waves (Written, 4 points)

In an interval $[0, L]$ with periodic boundary conditions, plane waves are given by

$$\psi_n = \frac{1}{\sqrt{L}} \exp\left(\frac{i}{\hbar} p_n x\right), \quad \text{where} \quad p_n = \frac{2\pi\hbar n}{L}. \tag{1}$$

a) Show that they form a complete set of orthonormal basis functions. In other words, show that:

$$\int_0^L dx\, \psi_n^*(x)\psi_m(x) = \delta_{n,m} \qquad \text{(orthonormality)}, \tag{2}$$

$$\sum_{n=-\infty}^{\infty} \psi_n^*(x)\psi_n(x') = \delta(x - x') \qquad \text{(completeness)}. \tag{3}$$

b) In the limit $L \to \infty$, the orthogonality and completeness relations read

$$\int_{-\infty}^{\infty} dx\, \exp\left[-\frac{i}{\hbar}(p - p')x\right] = 2\pi\hbar\, \delta(p - p') \quad \text{(orthogonality)}, \tag{4}$$

$$\int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar} \exp\left[\frac{i}{\hbar}(x - x')p\right] = \delta(x - x') \quad \text{(completeness)}. \tag{5}$$

Perform the limit $L \to \infty$ explicitly for ψ_n by using an appropriate prefactor and show both the orthogonality (4) and the completeness (5). (Note that the basis functions are no longer normalizable and are thus called a *general set of basis functions*.)

Exercise 2: Classical action (Written, 5 points)

In classical mechanics, the action S is defined as

$$S[q](t) = \int_{t_0}^{t} dt' \, L(\dot{q}(t'), q(t'), t') \,, \tag{6}$$

where L is the *Lagrangian function* (also called *Lagrangian*) and q is some generalized coordinate of the system.

a) Calculate the action $S(q_\tau, \tau)$ along classical paths for the following systems:
   - free particle,
   - harmonic oscillator with $V(q) = \frac{1}{2} m\omega^2 q^2$,
   - constant force F,

   where $q(t = \tau_0) = 0$ is the starting point and $q(t = \tau) = q_\tau$ is the endpoint of the path at some time $\tau > \tau_0$.

b) For classical paths with a fixed starting point $q(\tau_0) = q_0$, one can interpret S as a function of q_τ and τ, i.e. $S = S(q_\tau, \tau)$. Show that

$$\frac{\partial S}{\partial q_\tau} = p_\tau \,, \quad \frac{\partial S}{\partial \tau} = -H \,, \tag{7}$$

   where H is the Hamiltonian function of the system.

   Hint: For some fixed starting point q_0, each path can be parametrized by some arbitrary point q_τ on that path such that $q(t)$ itself can be thought of as a function of both q_τ and some general time t.

Exercise 3: Free propagator (Oral)

The propagator $K(x, x', t)$ of a Hamiltonian operator H (some differential operator in x) is defined as the solution of the Schrödinger equation

$$i\hbar \partial_t K(x, x', t) = H K(x, x', t) \tag{8}$$

with the initial condition $K(x, x', 0) = \delta(x - x')$.

a) Show that for the initial condition $\psi(x, t = 0) = \psi_0(x)$ the solution of the Schrödinger equation is given by

$$\psi(x, t) = \int dx' \, K(x, x', t) \psi_0(x') \,. \tag{9}$$

b) Show using Fourier transformation (plane wave expansion) that the propagator of a free particle with Hamiltonian operator $H = -\hbar^2 \partial_x^2 / 2m$ is given by

$$K(x, x', t) = \sqrt{\frac{m}{2\pi\hbar i t}} \exp\left(\frac{im(x - x')^2}{2\hbar t} \right) \,. \tag{10}$$