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Exercise 1: Fourier transform (Written, 6 points)

Given a complex-valued function $f : \mathbb{R} \to \mathbb{C}$, the one-dimensional Fourier transformation is defined as

$$\mathcal{F}[f(x)] = \int_{-\infty}^{\infty} \mathrm{d}x \; e^{-ikx} f(x) \equiv \hat{f}(k) \; . \tag{1}$$

a) Show that

$$\mathcal{F}[1] = 2\pi\delta(k) \quad \text{and} \quad \mathcal{F}[\delta(x)] = 1 \; , \tag{2}$$

where $\delta(q)$ is the Dirac delta function. Use $\int_{-\infty}^{\infty} \mathrm{d}q \; h(q)\delta(q) = h(0)$.

b) Show that the inverse Fourier transformation is

$$\mathcal{F}^{-1}[\hat{f}(k)] = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \; e^{ikx} \hat{f}(k) = f(x) \; . \tag{3}$$

c) Show that

$$\mathcal{F}[f(x+a)] = e^{ika} \; \mathcal{F}[f(x)] \; . \tag{4}$$

d) Show that

$$\mathcal{F}[\partial_x f(x)] = ik \; \mathcal{F}[f(x)] \; . \tag{5}$$

e) The convolution of two functions $f : \mathbb{R} \to \mathbb{C}$ and $g : \mathbb{R} \to \mathbb{C}$ is defined as

$$(f * g)(x) = \int_{-\infty}^{\infty} \mathrm{d}\xi \; f(\xi) g(x-\xi) = \int_{-\infty}^{\infty} \mathrm{d}\xi \; f(x-\xi) g(\xi) \; . \tag{6}$$

Show that

$$\mathcal{F}[(f * g)(x)] = \mathcal{F}[f(x)] \cdot \mathcal{F}[g(x)] \; . \tag{7}$$

f) Calculate the Fourier transformation of the Gaussian function

$$f(x) = e^{-ax^2} \quad \text{with} \quad a \neq 0 \; , \; \mathrm{Re}(a) \geq 0 \; , \; a \in \mathbb{C} \; . \tag{8}$$

Pay particular attention to the calculation for a purely imaginary a.

Exercise 2: Wave packet (Written, 6 points)

We consider a free particle of mass m. Its dispersion relation is $\omega(k) = \frac{\hbar}{2m}k^2$. As shown in the lecture, the functions $\psi_k(x) = e^{i(kx-\omega(k)t)}$ form a basis for the solution space of the Schrödinger equation for the free particle.

One solution of the Schrödinger equation for the free particle is the Gaussian wave packet. At time $t = 0$, the wave packet is of the form

$$\psi(x,0) = A\, e^{ik_0x} e^{-\frac{(x-x_0)^2}{4\sigma}} \; . \tag{9}$$

a) Calculate the normalization constant A.

b) Show that the wave packet becomes

$$\psi(x,t) = \left(\frac{\sigma}{2\pi\sigma_t^2}\right)^{\frac{1}{4}} e^{ik_0x} e^{-i\frac{\hbar}{2m}k_0^2 t} e^{-\frac{(x-(x_0+\hbar k_0 t/m))^2}{4\sigma_t}} \; , \quad \sigma_t = \sigma + i\frac{\hbar}{2m}t \tag{10}$$

for arbitrary time t.

c) Determine the velocity of the particle which is described by the wave packet. Use the equations

$$\langle v\rangle = \partial_t\langle x\rangle \; , \quad \langle x\rangle = \int_{-\infty}^{\infty} \mathrm{d}x\, \psi^*(x,t)x\psi(x,t) \; . \tag{11}$$

d) The uncertainty Δx is defined by $(\Delta x)^2 = \langle x^2\rangle - \langle x\rangle^2$. It is a measure of the width of a probability distribution. At $t = 0$, the uncertainty of the particle's position is given by $(\Delta x)^2|_{t=0} = \sigma$. Show that

$$(\Delta x)^2 = \sigma(a_0 + a_1 t^2) \tag{12}$$

for arbitrary time t.

e) A temperature profile $T(x,t)$ is described by the heat equation

$$\partial_t T(x,t) - D\partial_x^2 T(x,t) = 0 \; , \tag{13}$$

where D is a real constant. We consider a Gaussian temperature profile with $(\Delta x)^2|_{t=0} = \sigma$ at $t = 0$. Show that

$$(\Delta x)^2 = \sigma(b_0 + b_1 t) \tag{14}$$

for arbitrary time t. Take into account that $\langle x\rangle$ is defined by

$$\langle x\rangle = \int_{-\infty}^{\infty} \mathrm{d}x\, T(x,t)x \; . \tag{15}$$

in the context of the heat equation.

The solution of the heat equation can be obtained from the solution of the Schrödinger equation through Wick rotation $t \to -it$. In addition, we must set $k_0 = 0$ inside the solution of the Schrödinger equation since temperature distributions solely undergo diffusion.

Remark: Note the difference between equation 14 and 12, showing that the i inside the Schrödinger equation causes the quadratic broadening.

f) A linear dispersion relation $\omega(k) = c_0 + c_1 k$ would have changed the results for the free particle. Show that the wave packet would not have broadened in time.

Exercise 3: Double-slit experiment and uncertainty principle (Oral)

Electrons of momentum p_0 move perpendicularly towards an aperture with two slits of distance a. After passing the aperture, the electrons yield an interference pattern on a screen mounted parallel to the aperture. The distance between the screen and the aperture is d. The angles θ_1, θ_2 are enclosed by the normal of the screen and the electrons' path drawn from slit 1, 2 to the impact point on the screen.

a) Calculate the positions of the interference maxima on the screen. The probability to find electrons on the screen is

$$P \propto |\phi_1 + \phi_2|^2 \ , \tag{16}$$

where ϕ_i is the collected phase which can be calculated via

$$\phi_i = e^{\frac{i}{\hbar} S[x_i(t)]} \quad \text{with} \quad S[x_i(t)] = \int \mathrm{d}t \ L[x_i(t)] = \int \mathrm{d}t \ \frac{m}{2}(\partial_t x_i(t))^2 \ . \tag{17}$$

Here, $x_i(t)$ denotes the electrons' path that passes slit i. The angles θ_1 and θ_2 can be assumed to be small and treated as equal.

b) In this subtask, we consider a measurement device attached to the slits. The measurement device reveals the slit an electron passes before it hits the screen by the following principle: We treat the scattering of the electrons at the double-slit in a classical picture. The momentum δp transferred on the electron depends on the slit the electron went through. In case of elastic scattering, the momentum transfer is

$$\delta p_i = -p_0 \sin\theta_i \tag{18}$$

for slit i (here, we must not treat θ_1 and θ_2 as equal). If we try to measure δp, we have to take into account that the measurement device itself obeys the uncertainty principle

$$\Delta p \Delta x \geq h. \tag{19}$$

The uncertainty Δp of the momentum of the measurement device must fulfill the condition $\Delta p < |\delta p_2 - \delta p_1|$ in order to be able to discriminate between the two slits. Estimate the uncertainty Δx of the position of the measurement device.

c) Compare the uncertainty of the position of the measurement device with the distance of the interference maxima.