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Exercise 1: Derivation of Planck's law — Einstein's way (Oral)

Here we derive Planck's law along the lines of the seminal paper *"Zur Quantentheorie der Strahlung"* by A. Einstein[1].

- We consider a cavity with electromagnetic radiation of spectral energy density $u(\omega, T)$ where ω is the frequency and T the temperature.
- In addition, the cavity is populated with molecules with internal states of energies E_n (e.g., rotational and vibrational modes). In thermal equilibrium, the probability to find a molecule excited to E_n is given by the Boltzmann weight $P_n = Z^{-1} e^{-E_n/k_B T}$ with the Boltzmann constant k_B and the normalizing factor $Z = \sum_n e^{-E_n/k_B T}$ known as *partition function*.
- The rate of molecules that make a transition from level E_n to E_m by emission $(E_n > E_m)$ or absorption $(E_m > E_n)$ of a photon into or from the cavity is given by

$$W_{nm} = P_n \cdot \begin{cases} B_{nm} u(\omega_{nm}, T) + A_{nm} & \text{for} \quad E_n > E_m \quad \text{(Emission)} \\ B_{nm} u(\omega_{mn}, T) & \text{for} \quad E_n < E_m \quad \text{(Absorption)} \end{cases} \tag{1}$$

  where the photon must match the energy difference: $\hbar\omega_{nm} = E_n - E_m$. $B_{nm} u(\omega_{nm})$ and A_{nm} are the transition rates (probability for a transition per timestep) for radiation-induced emission/absorption and spontaneous emission, respectively.

Thermal equilibrium is characterized by $W_{nm} = W_{mn}$. Use this relation to derive an expression for the spectral energy density $u(\omega, T)$. It's asymptotic form for $T \to \infty$ is known to be the Rayleigh-Jeans law:

$$u(\omega, T) = \frac{k_B T}{\pi^2 c^3} \omega^2 \tag{2}$$

Use this knowledge to show that $u(\omega, T)$ takes the well-known form of Planck's law.

Exercise 2: Bohr-Sommerfeld Quantization (Oral)

Here we use a quantization procedure known as *Bohr-Sommerfeld quantization* to translate a classical theory into the quantum realm. This method historically preceded both the axioms of canonical quantization used in the lecture and the quantization by path integrals. It never gave rise to a complete, consistent quantum theory, even though it allows one to come to the correct conclusions in some cases.

[1] Phys. Z. 18, p. 121-128 (1917)

a) The Bohr-Sommerfeld quantization condition claims that, for a given Hamiltonian *function* $H(q_i, p_i)$ with D degrees of freedom $q_1, \ldots, q_D$, classical *periodic* solutions $(\mathbf{q}(t), \mathbf{p}(t))$ are allowed only if for a single cycle and for all $i = 1, \ldots, D$

$$I_{n_i} := \frac{1}{2\pi} \oint \mathrm{d}q_i \, p_i \stackrel{!}{=} \hbar(n_i + \alpha_i) \quad \text{where} \quad n_i \in \mathbb{N}_0 \tag{3}$$

and $\alpha_i \in \mathbb{R}$ are undetermined constants. Apply this condition on the one- and two-dimensional harmonic oscillator,

$$H_{1\mathrm{D}} = \frac{p^2}{2m} + \frac{m\omega^2}{2} q^2 \quad \text{and} \quad H_{2\mathrm{D}} = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{m\omega^2}{2}(q_x^2 + q_y^2)\,, \tag{4}$$

and derive the quantized energy levels E_n for both systems. In two dimensions, treat each dimension, (p_x, q_x) and (p_y, q_y), separately and discuss the degeneracy of energy levels.

b) We now focus on an electron orbiting in a homogeneous magnetic field $\mathbf{B} = \nabla \times \mathbf{A} = B\mathbf{e}_z$ in the context of Bohr-Sommerfeld quantization. Its Hamiltonian reads

$$H = \frac{\mathbf{\Pi}^2}{2m} \quad \text{with} \quad \mathbf{\Pi} = \mathbf{p} + \frac{e}{c}\mathbf{A} \tag{5}$$

Here, e is the elementary charge, c the speed of light, and $\mathbf{A}$ the vector potential. $\mathbf{\Pi} = m\dot{\mathbf{q}}$ is the kinematical momentum whereas $\mathbf{p}$ is the canonical momentum conjugate to the electron's position $\mathbf{q}$.

To simplify the problem, we employ our intuition that due to the Lorentz force, the electron periodically follows a circular trajectory of radius R in the x-y-plain. If we choose the origin such that it coincides with the electron's orbital center, we can write $\mathbf{q} = R\mathbf{e}_r$ and $\mathbf{p} = p_\varphi \mathbf{e}_\varphi$ in polar coordinates of the x-y-plain.

Then the Hamiltonian takes the simpler form

$$H = \frac{\Pi_\varphi^2}{2m} \quad \text{with} \quad \Pi_\varphi = p_\varphi + \frac{e}{c}\mathbf{A} \cdot \mathbf{e}_\varphi\,. \tag{6}$$

Here we choose a gauge $\mathbf{A} \propto \mathbf{e}_\varphi$ for the sake of convenience (what is its exact form in dependence of B?)

According to the Bohr-Sommerfeld prescription, the canonical momentum p_φ is required to obey Eq. (3), i.e.,

$$I_{n_\varphi} = \frac{R}{2\pi} \int_0^{2\pi} \mathrm{d}\varphi \, p_\varphi \stackrel{!}{=} \hbar(n_\varphi + \alpha_\varphi)\,. \tag{7}$$

How does the (quantized) kinetic energy $E_{\mathrm{kin}} = \Pi_\varphi^2/2m$ depend on the magnetic flux ϕ enclosed by the electron's trajectory? Show that the flux is now quantized as well, namely in units of flux quanta $\Phi_0 \equiv hc/e$ with $\hbar = h/2\pi$.

Exercise 3: Commutators (Written, 5 points)

In this exercise, you will derive several operator identities — the bread and butter tools of any quantum physicist. The commutator of linear operators $A, B \in L(\mathcal{H})$ acting on some Hilbert space $\mathcal{H}$ is defined as

$$[A, B] := AB - BA\,. \tag{8}$$

a) Let A,B, and C be linear operators on $\mathcal{H}$. Show that

$$[AB, C] = A\,[B, C] + [A, C]\,B \tag{9a}$$
$$[A, BC] = [A, B]\,C + B\,[A, C]\;. \tag{9b}$$

b) Let x and $p = -i\hbar\partial_x$ be the position and momentum operator, as introduced in the lecture. Evaluate the commutators

$$[x, p]\;, \quad \left[x, p^2\right]\;, \quad \left[x^2, p^2\right]\;, \quad \left[xp, p^2\right]\;. \tag{10}$$

c) Now let f and g be smooth functions with a Taylor series representation on $\mathbb{R}$. Show that

$$[p, g(x)] = -i\hbar\frac{\mathrm{d}g(x)}{\mathrm{d}x} \quad \text{and} \quad [x, f(p)] = i\hbar\frac{\mathrm{d}f(p)}{\mathrm{d}p} \tag{11}$$

where $f(A) \equiv \sum_{n=0}^{\infty} f_n\, A^n$ with an arbitrary operator A and Taylor coefficients f_n.