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Exercise 1: Baker-Campbell-Hausdorff formula (Written, 2 points)

Consider two non-commuting operators A and B, for which the relations $[A,[A,B]] = [B,[A,B]] = 0$ are satisfied.

a) Show that for the operators A and B, the equality

$$e^{-At}Be^{At} = B - t\,[A,B] \tag{1}$$

is satisfied, where t is an arbitrary number.

b) Show that both operators satisfy the Baker-Campbell-Hausdorff formula

$$e^{A+B} = e^A e^B e^{-\frac{[A,B]}{2}}. \tag{2}$$

Hint: Derive a first-order differential equation for the operator $W(t) = e^{-tA}e^{t(A+B)}$ and solve it.

Exercise 2: Heisenberg uncertainty relations (Oral)

a) Show that for the momentum and position operators, p and x respectively, the uncertainty principle

$$\Delta p \cdot \Delta x \geqslant \frac{\hbar}{2} \tag{3}$$

applies, where we have defined the *standard deviation* $\Delta A = \sqrt{\langle (A - \langle A\rangle)^2\rangle}$ of an operator A of expectation value $\langle A\rangle$.

b) Given two hermitian operators A and B, prove the generalized uncertainty principle

$$(\Delta A)^2 \cdot (\Delta B)^2 \geqslant \frac{\langle \mathrm{i}\,[A,B]\rangle^2}{4}. \tag{4}$$

Show that this result is consistent with task a).

Exercise 3: Continuity and unitarity (Oral)

a) Starting from the Schrödinger equation and the probability current density,

$$\mathbf{j}(\mathbf{r},t) = -\frac{\mathrm{i}\hbar}{2m}\left[\psi^*\nabla_{\mathbf{r}}\psi(\mathbf{r},t) - \psi(\mathbf{r},t)\nabla_{\mathbf{r}}\psi^*(\mathbf{r},t)\right], \tag{5}$$

show that the probability density, $|\psi(\mathbf{r},t)|^2$, is conserved.

b) The time evolution of a pure state is described by

$$|\psi(t)\rangle = U(t, t_0)\,|\psi(t_0)\rangle\,, \tag{6}$$

where $U(t, t_0)$ is the time-evolution operator. Show which four properties this operator must satisfy by using, for example, subtask a). Show that these are satisfied for an operator of the form

$$U(t, t_0) = e^{-\frac{\mathrm{i}}{\hbar}H(t-t_0)}, \tag{7}$$

where H is the Hamiltonian.