Exercise 1: Creation and annihilation operators (Written, 3 points)

Given the 2-dimensional harmonic oscillator Hamiltonian:

$$H = \hbar \omega_+ (a_+^\dagger a_+ + \frac{1}{2}) + \hbar \omega_- (a_-^\dagger a_- + \frac{1}{2});$$  \hspace{1cm} (1)

where the creation and annihilation operators $a_\pm^\dagger$, $a_\pm$ satisfy the following commutation rules:

$$[a_\pm^\dagger, a_\pm] = 1; \quad [a_\pm, a_\pm^\dagger] = [a_\pm^\dagger, a_\pm^\dagger] = 0;$$  \hspace{1cm} (2)

$$[a_\pm, a_\mp] = [a_\pm^\dagger, a_\mp^\dagger] = 0;$$  \hspace{1cm} (3)

a) Show that the Hamiltonian of the system is diagonal with respect to the eigenstates of the number operators $N_+ = a_+^\dagger a_+$ and $N_- = a_-^\dagger a_-$ and find the respective eigenenergies. Does the measurement of the observable $N = \langle N_+ + N_- \rangle$ specify the state of the system?

b) Define the groundstate of the system $|0,0\rangle$ through $a_\pm |0,0\rangle = 0$, correctly normalized $\langle 0|0 \rangle = 1$. The Hilbert space can be constructed by the application of creation operators on $|0,0\rangle$. Show that the commutation relations

$$[N_+, a_+^\dagger] = a_+^\dagger \quad [N_+, a_+] = -a_+$$  \hspace{1cm} (4)

hold (same for $N_- \). Find the normalized eigenvectors $|n_+,n_-\rangle$ of both number operators with eigenvalues $n_+$ and $n_-$. 

c) Verify the following relations:

$$a_+^\dagger |n_+,n_-\rangle = \sqrt{n_+ + 1} |n_+ + 1,n_-\rangle$$  \hspace{1cm} (5)

$$a_+ |n_+,n_-\rangle = \sqrt{n_+-1} |n_+-1,n_-\rangle$$  \hspace{1cm} (6)

Exercise 2: Coherent states (Oral)

For every $\alpha \in \mathbb{C}$, we define the coherent state

$$|\psi_\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$  \hspace{1cm} (7)

being $|n\rangle$ an eigenstate of the 1-dimensional harmonic oscillator Hamiltonian.
a) Show that

\[ |\psi_\alpha\rangle = e^{\alpha a^\dagger - \alpha^* a} |0\rangle \] (8)

where \( a^\dagger \) and \( a \) are creation and annihilation operators, respectively. (Tip: \( e^{-\alpha a} |0\rangle = |0\rangle \))

b) Calculate \( \langle x \rangle_\alpha = \langle \psi_\alpha | x | \psi_\alpha \rangle \), \( \langle p \rangle_\alpha \), \( \Delta x_\alpha \), \( \Delta p_\alpha \) and show that, for all \( \alpha \in \mathbb{C} \),

\[ \Delta x_\alpha \Delta p_\alpha = \frac{\hbar}{2} \] (9)

is valid, i.e., coherent states minimize the position and momentum uncertainty relation. (Tip: \( a |\psi_\alpha\rangle = \alpha |\psi_\alpha\rangle \))

c) Show that coherent states are not orthogonal and the relation

\[ \langle \psi_\alpha | \psi_\beta \rangle = e^{-\frac{1}{2}(|\alpha|^2 + |\beta|^2 - 2\alpha^* \beta)} \] (10)

is valid.

d) Derive the time evolution of coherent state \( |\psi_\alpha(t)\rangle \) and of the expectation values \( \langle x \rangle_t \) and \( \langle p \rangle_t \) under the Hamiltonian \( H = \hbar \omega (a^\dagger a + \frac{1}{2}) \).

Exercise 3: Wave packet at a potential barrier (Oral)

Consider the following 1-dimensional potential

\[ V(x) = \begin{cases} 
0 & x < 0 \\
V_0 & 0 \leq x \leq a \\
0 & x > a 
\end{cases} \] (11)

with \( V_0 > 0 \), and the solution of stationary Schrödinger equation

\[ \psi_k(x) = \begin{cases} 
e^{ikx} + r(k)e^{-ikx} & x < 0 \\
B(k)e^{-kx} + B'(k)e^{kx} & 0 \leq x \leq a \\
t(k)e^{ikx} & x > a 
\end{cases} \] (12)

with \( k = \sqrt{2mE}/\hbar \), \( \bar{k}^2 = 2mV_0 - k^2 \).

Suppose that at time \( t_0 \) one has a wave packet \( \psi(x, t_0) \) centered at \(-L\) and far from the barrier \( L \gg a \) with \( \sigma \ll L \)

\[ \psi(x, t_0) = \frac{1}{(2\pi\sigma)^{\frac{1}{2}}} e^{ik_0x} e^{-\frac{(x+L)^2}{4\sigma}} \] (13)

with \( k_0 < \bar{k} \).
a) Decompose the wave packet $\psi(x, t_0)$ into eigenstates of the Hamiltonian.

b) Suppose that the coefficients $t(k), r(k), B(k), B'(k)$ are $k$-independent. Calculate the time evolution of the wave packet $\psi(x, t)$ and show that, after reaching the barrier, it splits into two wave packets: reflected and transmitted.

c) In the approximations of the previous subtask, calculate the time evolution of the center positions of the incident $\langle x \rangle_i$, transmitted $\langle x \rangle_t$ and reflected $\langle x \rangle_r$ wave packets.

d)* Use the computer to solve numerically the time evolution of the wave packet in the general case with $t(k), r(k), B(k), B'(k)$ explicitly $k$-dependent.