Prof. Dr. Hans Peter Büchler
Institut für Theoretische Physik III, Universität Stuttgart

29. November 2016
WS 2016/17

Exercise 1: Creation and annihilation operators (Written, 3 points)

Given the 2-dimensional harmonic oscillator Hamiltonian:

$$H = \hbar\omega_+(a_+^\dagger a_+ + \frac{1}{2}) + \hbar\omega_-(a_-^\dagger a_- + \frac{1}{2}); \tag{1}$$

where the creation and annihilation operators $a_\pm^\dagger$, $a_\pm$ satisfy the following commutation rules:

$$\left[a_\pm, a_\pm^\dagger\right] = 1; \qquad [a_\pm, a_\pm] = \left[a_\pm^\dagger, a_\pm^\dagger\right] = 0; \tag{2}$$

$$\left[a_\pm, a_\mp^\dagger\right] = [a_\pm, a_\mp] = \left[a_\pm^\dagger, a_\mp^\dagger\right] = 0; \tag{3}$$

a) Show that the Hamiltonian of the system is diagonal with respect to the eigenstates of the number operators $N_+ = a_+^\dagger a_+$ and $N_- = a_-^\dagger a_-$ and find the respective eigenenergies. Does the measurement of the observable $N = \langle N_+ + N_- \rangle$ specify the state of the system?

b) Define the groundstate of the system $|0,0\rangle$ through $a_\pm |0,0\rangle = 0$, correctly normalized $\langle 0,0|0,0\rangle = 1$. The Hilbert space can be constructed by the application of creation operators on $|0,0\rangle$. Show that the commutation relations

$$[N_+, a_+] = -a_+ \qquad \left[N_+, a_+^\dagger\right] = a_+^\dagger \tag{4}$$

hold (same for N_-). Find the normalized eigenvectors $|n_+, n_-\rangle$ of both number operators with eigenvalues n_+ and n_-.

c) Verify the following relations:

$$a_+^\dagger |n_+, n_-\rangle = \sqrt{n_+ + 1}\, |n_+ + 1, n_-\rangle \tag{5}$$

$$a_+ |n_+, n_-\rangle = \sqrt{n_+}\, |n_+ - 1, n_-\rangle \tag{6}$$

Exercise 2: Coherent states (Oral)

For every $\alpha \in \mathbb{C}$, we define the coherent state

$$|\psi_\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n\geq 0} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \tag{7}$$

being $|n\rangle$ an eigenstate of the 1-dimensional harmonic oscillator Hamiltonian.

a) Show that

$$|\psi_\alpha\rangle = e^{\alpha a^\dagger - \alpha^* a}\,|0\rangle \tag{8}$$

where $a^\dagger$ and a are creation and annihilation operators, respectively.
(Tip: $e^{-\alpha a}\,|0\rangle = |0\rangle$)

b) Calculate $\langle x\rangle_\alpha = \langle\psi_\alpha\,|x|\,\psi_\alpha\rangle$, $\langle p\rangle_\alpha$, Δx_α, Δp_α and show that, for all $\alpha \in \mathbb{C}$,

$$\Delta x_\alpha \Delta p_\alpha = \frac{\hbar}{2} \tag{9}$$

is valid, *i.e.*, coherent states minimize the position and momentum uncertanty relation.
(Tip: $a\,|\psi_\alpha\rangle = \alpha\,|\psi_\alpha\rangle$)

c) Show that coherent states are not orthogonal and the relation

$$\langle\psi_\alpha|\psi_\beta\rangle = e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2-2\alpha^*\beta)} \tag{10}$$

is valid.

d) Derive the time evolution of coherent state $|\psi_\alpha(t)\rangle$ and of the expectation values $\langle x\rangle_t$ and $\langle p\rangle_t$ under the Hamiltonian $H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$.

Exercise 3: Wave packet at a potential barrier (Oral)

Consider the following 1-dimensional potential

$$V(x) = \begin{cases} 0 & x < 0 \\ V_0 & 0 \le x \le a \\ 0 & x > a \end{cases} \tag{11}$$

with $V_0 > 0$, and the solution of stationary Schrödinger equation

$$\psi_k(x) = \begin{cases} e^{ikx} + r(k)e^{-ikx} & x < 0 \\ B(k)e^{-\bar{k}x} + B'(k)e^{\bar{k}x} & 0 \le x \le a \\ t(k)e^{ikx} & x > a \end{cases} \tag{12}$$

with $k = \sqrt{2mE}/\hbar$, $\bar{k}^2 = 2mV_0 - k^2$.

Suppose that at time t_0 one has a wave packet $\psi(x,t_0)$ centered at $-L$ and far from the barrier $L \gg a$ with $\sigma \ll L$

$$\psi(x,t_0) = \frac{1}{(2\pi\sigma)^{\frac{1}{4}}}e^{ik_0x}e^{-\frac{(x+L)^2}{4\sigma}} \tag{13}$$

with $k_0 < \bar{k}$.

a) Decompose the wave packet $\psi(x, t_0)$ into eigenstates of the Hamiltonian.

b) Suppose that the coefficients $t(k)$, $r(k)$, $B(k)$, $B'(k)$ are k-independent. Calculate the time evolution of the wave packet $\psi(x, t)$ and show that, after reaching the barrier, it splits into two wave packets: reflected and transmitted.

c) In the approximations of the previous subtask, calculate the time evolution of the center positions of the incident $\langle x \rangle_i$, transmitted $\langle x \rangle_t$ and reflected $\langle x \rangle_r$ wave packets.

d)* Use the computer to solve numerically the time evolution of the wave packet in the general case with $t(k)$, $r(k)$, $B(k)$, $B'(k)$ explicitely k-dependent.