Prof. Dr. Hans Peter Büchler
Institut für Theoretische Physik III, Universität Stuttgart

6. Dezember 2016
WS 2016/17

Exercise 1: Linear algebra basics (Written, 6 points)

Let $\mathcal{H}$ be a finite-dimensional Hilbert space whose elements are denoted by $|\psi\rangle$, $|\phi\rangle$, ... Further, we consider linear operators A, B on that Hilbert space.

a) The *Hermitian conjugate* (or *adjoint*) of A (also denoted $A^\dagger$) is defined as

$$\langle\psi|A^\dagger\phi\rangle = \langle A\psi|\phi\rangle = \langle\phi|A\psi\rangle^* \,. \tag{1}$$

Show that $(AB)^\dagger = B^\dagger A^\dagger$.

b) An operator satisfying $A = A^\dagger$ is called *Hermitian* or *self-adjoint*. Show that the eigenvalues of Hermitian operators are real and the corresponding eigenvectors of two different eigenvalues are orthogonal. (*Remark*: You can assume that there are no degenerate eigenvalues.)

c) A (linear) operator that satisfies $UU^\dagger = U^\dagger U = I$ with I the identity, or equivalently $U^{-1} = U^\dagger$, is called a *unitary* operator. Show that unitary operators leave the norm of a vector unchanged. Show that the eigenvalues λ_n of a unitary operator have modulus unity, i.e. $\lambda_n = e^{i\phi_n}$ with $\phi_n \in \mathbb{R}$, and that eigenvectors corresponding to different eigenvalues are orthogonal.

d) Let A be a Hermitian operator. Show that the operator $U = e^{i\alpha A}$, $\alpha \in \mathbb{R}$, is unitary.

e) Consider an orthonormal basis set $\{|n\rangle\}$ and another basis set $\{|n'\rangle\}$ with $|n'\rangle = U\,|n\rangle$. Show that $\{|n'\rangle\}$ is also orthonormal. If we denote the matrix elements of an operator A by $A_{mn} = \langle m|A|n\rangle$, how do the matrix elements A_{mn} relate to the matrix elements in the basis $\{|n'\rangle\}$?

f) Let A and B be Hermitian operators with $[A, B] = 0$. Show that A and B can be diagonalized simultaneously. (*Remark:* Neglect the case of degenerate eigenvalues.)

Exercise 2: Position space and momentum space representation (Oral)

Consider the position space basis $\{|x\rangle\}$ which is an eigenbasis of the position operator $\hat{x}$ with eigenvalues x and the momentum space basis $\{|p\rangle\}$ which is an eigenbasis of the momentum operator $\hat{p}$ with eigenvalues p. The spectra of both operators are continuous such that the normalization condition and the completeness relation read

$$\langle\lambda'|\lambda\rangle = \delta(\lambda' - \lambda)\,, \qquad I = \int d\lambda\; |\lambda\rangle\,\langle\lambda| \qquad \lambda = x, p\,. \tag{2}$$

a) Using the canonical commutation relation, show that $e^{\frac{i}{\hbar}\hat{p}a}\hat{x}e^{-\frac{i}{\hbar}\hat{p}a} = \hat{x} + aI$. Show that $e^{-\frac{i}{\hbar}\hat{p}a}\,|x\rangle$ is an eigenstate of $\hat{x}$ with eigenvalue $x + a$, i.e. $e^{-\frac{i}{\hbar}\hat{p}a}\,|x\rangle = |x + a\rangle$.

b) Let $\langle x|\phi\rangle = \phi(x)$ be the component of a physical state vector $|\phi\rangle$ in the basis $|x\rangle$. Calculate the matrix elements of the operators $\hat{x}$ and $e^{-\frac{i}{\hbar}\hat{p}a}$ in the position space basis and deduce the representation of the momentum operator in position space.

c) Determine the wave functions $\psi_p(x)$ of the eigenvectors $|p\rangle$ in the position space representation. (*Hint:* Choose the normalization factor such that (2) is fulfilled.)

d) Let $\langle p|\phi\rangle = \tilde{\phi}(p)$ be the component of a physical state vector $|\phi\rangle$ in the basis $|p\rangle$. Find the action of the operators $\hat{x}$ and $\hat{p}$ in the momentum space representation.

e) Show that $e^{\frac{i}{\hbar}\hat{p}a} f(\hat{x}) e^{-\frac{i}{\hbar}\hat{p}a} = f(\hat{x} + aI)$ and $e^{-a\partial_x} f(x) = f(x-a)$.

Exercise 3: Schrödinger equation in position and momentum space (Written, 3 points)

a) Show that the time evolution of a state $|\psi\rangle$ satisfying the Schrödinger equation with a time-independent Hamiltonian $\hat{H}$ is given by the time evolution operator $\hat{U}(t) = e^{-\frac{i}{\hbar}\hat{H}t}$.

b) Consider a Hamiltonian of the form $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$. Write down the Schrödinger equation in position and momentum space.

c) Consider a potential of the form $V(x) = g\delta(x)$ $(g < 0)$ in position space. Calculate the eigenvalue and the wave function of the bound state by solving the time-independent Schrödinger equation in momentum representation. (*Hint:* You may assume that the wave function decays fast enough in momentum space as well).

Exercise 4: Angular momentum (Oral)

Consider the angular momentum operator $\mathbf{L} = \mathbf{r} \wedge \mathbf{p}$. In the lecture it was shown that $\mathbf{L}$ is the infinitesimal generator of rotations such that rotations around some axis $\mathbf{n}$ with $\mathbf{n}^2 = 1$ about some angle ω can be written as $U_{\boldsymbol{\omega}} = \exp(-i\omega \mathbf{L}\cdot\mathbf{n}/\hbar)$.
If $U_{\boldsymbol{\omega}}$ is the operator performing a rotation around some axis $\boldsymbol{\omega} = \omega\mathbf{n}$ in the Hilbert space, i.e. $|\phi_{\boldsymbol{\omega}}\rangle = U_{\boldsymbol{\omega}}|\phi\rangle$; a *scalar* operator S transforms like

$$U_{\boldsymbol{\omega}}^\dagger\, S\, U_{\boldsymbol{\omega}} = S\,, \tag{3}$$

and a *vector* operator $\mathbf{X}$ transforms like

$$U_{\boldsymbol{\omega}}^\dagger\, \mathbf{X}\, U_{\boldsymbol{\omega}} = R_{\boldsymbol{\omega}}\, \mathbf{X}\,, \tag{4}$$

where $R_{\boldsymbol{\omega}}$ is the usual rotation matrix in three dimensions around some axis $\boldsymbol{\omega}$.

a) Show that for a scalar operator S, $[\mathbf{L}, S] = 0$.

b) Using that $\mathbf{r}$ and $\mathbf{p}$ are vector operators, show that $\mathbf{L}$ is also a vector operator. (*Hint:* Consider the components of $U_{\boldsymbol{\omega}}^\dagger\,\mathbf{r}\wedge\mathbf{p}\,U_{\boldsymbol{\omega}}$ and show that $U_{\boldsymbol{\omega}}^\dagger\,\mathbf{r}\wedge\mathbf{p}\,U_{\boldsymbol{\omega}} = U_{\boldsymbol{\omega}}^\dagger\,\mathbf{r}\,U_{\boldsymbol{\omega}} \wedge U_{\boldsymbol{\omega}}^\dagger\,\mathbf{p}\,U_{\boldsymbol{\omega}}$.)

c) Show that $[\mathbf{L}, \mathbf{p}\cdot\mathbf{r}] = 0$ on the one hand by explicitely calculating the commutator and on the other hand by showing that $\mathbf{p}\cdot\mathbf{r}$ is a scalar operator.