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Exercise 1: Representation of the angular momentum (Written, 5 points)

The purpose of this exercise is to work out the connection between the algebra of angular momentum (see QM Script, Chapter 4.3) and the algebra of the uncoupled, 2-dimensional harmonic oscillator (see Sheet 6, Exercise 1). We start by defining the momentum operators

$$L_z = \frac{\hbar}{2}(a_+^\dagger a_+ - a_-^\dagger a_-) \qquad L_+ = \hbar a_+^\dagger a_- \qquad L_- = \hbar a_-^\dagger a_+ \tag{1}$$

in terms of creation and annihilation operators $a_+^\dagger$, $a_-^\dagger$ and a_+, a_- that fulfill the harmonic oscillator commutation relations.

a) Prove that the angular momentum commutation relations are satisfied

$$\left[L_z, L_\pm\right] = \pm\hbar L_\pm \qquad \left[L_+, L_-\right] = 2\hbar L_z \; . \tag{2}$$

b) Prove that

$$L^2 = L_z^2 + \frac{1}{2}L_+L_- + \frac{1}{2}L_-L_+ = \frac{\hbar^2}{2}N\left(\frac{N}{2}+1\right) \; , \tag{3}$$

where $N = N_+ + N_-$ with the number operators $N_+ = a_+^\dagger a_+$ und $N_- = a_-^\dagger a_-$.

c) Show that the operators L_+, L_-, L_z, and L^2 act on the eigenstates $|n_+, n_-\rangle$ of the operators N_+ and N_- as follows:

$$\begin{aligned}
L_+ |n_+, n_-\rangle &= \hbar\sqrt{n_-(n_+ + 1)}\, |n_+ + 1, n_- - 1\rangle \\
L_- |n_+, n_-\rangle &= \hbar\sqrt{n_+(n_- + 1)}\, |n_+ - 1, n_- + 1\rangle \\
L_z |n_+, n_-\rangle &= \frac{\hbar}{2}(n_+ - n_-)\, |n_+, n_-\rangle \\
L^2 |n_+, n_-\rangle &= \frac{\hbar^2}{2}(n_+ + n_-)\left(\frac{n_+ + n_-}{2} + 1\right) |n_+, n_-\rangle \; .
\end{aligned} \tag{4}$$

Think of how the matrix representation of L_+, L_-, L_z, and L^2 in the basis $|n_+, n_-\rangle$ looks like. Which states $|n_+, n_-\rangle$ are coupled by the operators?

d) We define

$$l = \frac{1}{2}(n_+ + n_-) \qquad m = \frac{1}{2}(n_+ - n_-) \; . \tag{5}$$

Using these definitions, show that the equations (4) reduce to the familar expressions for the L_+, L_-, L_z, and L^2 operators:

$$\begin{aligned}
L_+ |l,m\rangle &= \hbar\sqrt{l(l+1)-m(m+1)}\,|l,m+1\rangle \\
L_- |l,m\rangle &= \hbar\sqrt{l(l+1)-m(m-1)}\,|l,m-1\rangle \\
L_z |l,m\rangle &= \hbar m\,|l,m\rangle \\
L^2 |l,m\rangle &= \hbar^2 l(l+1)\,|l,m\rangle\ .
\end{aligned} \tag{6}$$

e) Write $|l,m\rangle$ in terms of $a_+^\dagger$, $a_-^\dagger$, and the vacuum state $|0,0\rangle$. Interpret the obtained result.

Exercise 2: Symmetric top (Oral)

We study rotational states of molecules. The molecules can be considered to be rigid rotors, i.e. the distance of the atoms within the molecules can be assumed to be fixed. This exercise deals with molecules called symmetric tops in which two moments of inertia are the same ($I_x = I_y \equiv I_\perp$). The third moment of inertia ($I_z \equiv I_\parallel$) has a different value.

a) Express the Hamiltonian of such a system in terms of the angular momentum operators L^2 and L_z. Provide examples for molecules whose rotation can be well described by this Hamiltonian.

b) Determine the eigenvalues and eigenstates of the Hamiltonian.

Exercise 3: Asymmetric top (Oral)

In this exercise, we consider rotating molecules whose three moments of inertia (I_x, I_y, and I_z) have different values. The molecules are called asymmetric tops.

a) The Hamiltonian of such a system is

$$H = \frac{L_x^2}{2I_x} + \frac{L_y^2}{2I_y} + \frac{L_z^2}{2I_z}. \tag{7}$$

Show that the Hamiltonian commutes with L^2. Express the Hamiltonian in terms of L_z and the ladder operators $L_\pm = L_x \pm iL_y$.

b) We split the Hilbert space $\mathcal{H}$ of the system into subspaces $\mathcal{H}_l'$ and $\mathcal{H}_l''$, so that the sum $\bigoplus_l(\mathcal{H}_l' \oplus \mathcal{H}_l'')$ is the full Hilbert space again. The subspace $\mathcal{H}_l'$ contains the eigenstates $|l,m\rangle$ of L^2 and L_z with $m = l, l-2, \dots, -l+2, -l$. The subspace $\mathcal{H}_l''$ contains the eigenstates $|l,m\rangle$ with $m = l-1, l-3, \dots, -l+3, -l+1$.

Show that the Hamiltonian leaves the subspaces invariant, i.e. if the Hamiltonian acts on a state of a certain subspace, the resulting state belongs to the same subspace.

c) Consider the operator $U = \exp(-i\pi L_y/\hbar)$. Make use of $[U, H] = 0$ and $U\,|l, m\rangle = (-1)^{l-m}\,|l, -m\rangle$ in order to show that the full Hilbert space $\mathcal{H}$ can be partioned as

$$\mathcal{H} = \bigoplus_l \mathcal{H}_l \quad \text{with} \quad \mathcal{H}_l = \mathcal{H}'_{l+} \oplus \mathcal{H}'_{l-} \oplus \mathcal{H}''_{l+} \oplus \mathcal{H}''_{l-}\ , \tag{8}$$

where all of the subspaces are invariant under the Hamiltonian.

Hint: At first, convince yourself that the operator U commutes with L and that the operator U leaves the subspaces $\mathcal{H}'_l$ and $\mathcal{H}''_l$ invariant. Then, consider that the operator U couples the state $|l, m\rangle$ to the state $|l, -m\rangle$. The stated partition into subspaces follows from the diagonalization of the corresponding 2×2 matrices.

Supplementary question: What is the physical meaning of the operator U?

d) Determine the eigenvalues and eigenstates of the Hamiltonian for $l = 1$. Compare the results with Exercise 2.