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Exercise 1: Spherical oscillator (Written, 8 Pts.)

In this exercise we diagonalize the three dimensional, isotropic harmonic oscillator, given by the Hamiltonian

$$H_{3\mathrm{D}} = \frac{\mathbf{p}^2}{2m} + \frac{m\omega^2}{2}\mathbf{x}^2 \,, \tag{1}$$

in two ways. Here, $\mathbf{x}$ and $\mathbf{p}$ obey canonical commutation relations: $[x_i, p_j] = i\hbar\delta_{ij}$.

The simplest method to diagonalize $H_{3\mathrm{D}}$ is the following:

a) Show that $H_{3\mathrm{D}}$ can be decomposed into the sum of three one-dimensional harmonic oscillators. Thereby derive the eigenvectors and eigenvalues of $H_{3\mathrm{D}}$ and comment on the degeneracy of energy levels.

In the remainder of this exercise, we follow an alternative route to diagonalize $H_{3\mathrm{D}}$ in analogy to the Hydrogen atom (known from the lecture):

b) Show that $[\mathbf{L}^2, H] = 0 = [L_z, H]$. Similarly to the derivation of the Hydrogen atom, show that angular and radial dependence can be separated and write $H_{3\mathrm{D}}$ in spherical coordinates with angular momentum operators.

c) Make the ansatz

$$\Psi(\mathbf{r}) = \frac{x_l(r)}{r} Y_{lm}(\phi, \theta) \tag{2}$$

for the eigenfunctions and derive a differential equation for the radial component $x_l(r)$. Here, $Y_{lm}(\phi, \theta)$ are the spherical harmonics that were introduced in the lecture.

d) Define $\rho \equiv \sqrt{m\omega/\hbar}\, r$ and $\varepsilon \equiv 2E/\hbar\omega$ and show that the DGL takes the form

$$-\partial_\rho^2 x_l + \left[\frac{l(l+1)}{\rho^2} + \rho^2\right] x_l = \varepsilon\, x_l \,. \tag{3}$$

Consider the asymptotic behaviour of the latter for $\rho \to \infty$ and $\rho \to 0$ and show that this suggests the ansatz

$$x_l(\rho) = e^{-\rho^2/2}\, \rho^{l+1} \sum_{n=0}^{\infty} a_n \rho^n \,. \tag{4}$$

Show that the recursion relation for the coefficients a_n reads

$$a_{n+2} = \frac{2(n+l+1) - (\varepsilon - 1)}{(n+l+3)(n+l+2) - l(l+1)}\, a_n \tag{5}$$

and explain why $a_n = 0$ for odd n.

Finally, determine the eigenvalues of $H_{3\mathrm{D}}$ and comment on the degeneracy.

Compare your result with a).

Exercise 2: Runge-Lenz vector (Written, 6 Pts. + 2 Bonus Pts.)

In classical mechanics, Kepler problems[1] feature an additional conserved quantity ("integral of motion") called the *Runge-Lenz vector* $\mathbf{M}_{\mathrm{cl}}$. The latter lies within the plane of motion and is parallel to the major axis of the elliptical orbit. *Classically* one finds

$$\mathbf{M}_{\mathrm{cl}} = \frac{1}{me^2}\mathbf{L} \wedge \mathbf{P} + \frac{\mathbf{Q}}{Q} \qquad \text{with the Hamiltonian} \qquad H_{\mathrm{cl}} = \frac{\mathbf{P}^2}{2m} - \frac{e^2}{Q} \tag{6}$$

where $\mathbf{P}$ and $Q = \|\mathbf{Q}\|$ are relative momenta and coordinates of the two constituents. If the theory is quantized, $\mathbf{P}$ and $\mathbf{Q}$ become operators with canonical commutation relations: $[Q_i, P_j] = i\hbar\delta_{ij}$. This yields the Hamiltonian of the Hydrogen atom known from the lecture:

$$H = \frac{\mathbf{P}^2}{2m} - \frac{e^2}{Q} \tag{7}$$

There is, however, a subtlety in quantizing the Runge-Lenz vector $\mathbf{M}_{\mathrm{cl}}$: Simply replacing $\mathbf{P}$, $\mathbf{L}$ and $\mathbf{Q}$ by operators in $\mathbf{M}_{\mathrm{cl}}$ yields a non-hermitian operator, $\mathbf{M}_{\mathrm{cl}}^\dagger \neq \mathbf{M}_{\mathrm{cl}}$ (Why?). This clashes with the requirement of a *measurable* conserved quantity.

Thus one defines the symmetrized version

$$\mathbf{M} := \frac{1}{2me^2}\left(\mathbf{L} \wedge \mathbf{P} - \mathbf{P} \wedge \mathbf{L}\right) + \frac{\mathbf{Q}}{Q} \tag{8}$$

of the Runge-Lenz vector (operator) with $\mathbf{M}^\dagger = \mathbf{M}$ (Why?) and square

$$\mathbf{M}^2 = \frac{2}{me^4} H \left(\mathbf{L}^2 + \hbar^2 \mathbb{1}\right) + \mathbb{1}\,. \tag{9}$$

a) Show the following:

$$\mathbf{M} \cdot \mathbf{L} = 0 \tag{10a}$$

$$\left[\mathbf{L}^2, \mathbf{M}^2\right] = 0 \tag{10b}$$

$$\left[L_z, \mathbf{M}^2\right] = 0 \tag{10c}$$

$$[L_i, M_j] = i\hbar\,\varepsilon_{ijk} M_k \tag{10d}$$

$$[H, \mathbf{M}] = 0 \tag{10e}$$

In conclusion, we found that H, $\mathbf{L}^2$, L_z, and $\mathbf{M}^2$ define a set of pairwise commuting observables, i.e., there is a basis in which all four operators are diagonal.

Hints: The following may be useful:

- Write $(\mathbf{v} \wedge \mathbf{w})_i = \varepsilon_{ijk} v_j w_k$ in terms of the Levi-Civita symbol ε_{ijk}.
- Use the relation $\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$ where Einstein notation is used.

[1] Two bodies interacting via a radially symmetric inverse square-law force. For instance: planetary motion ruled by gravitation, the Hydrogen atom with the Coulomb force, etc..

- Use that $[L_i, V_j] = i\hbar\varepsilon_{ijk} V_k$ if $\mathbf{V}$ is a vector operator.
- Derive and use the commutator $[1/Q, P_k] = -i\hbar\frac{Q_k}{Q^3}$.

b*) Desperately looking for more commutators to evaluate? Then show that

$$[M_i, M_j] = -\frac{2i\hbar}{me^4}\,\varepsilon_{ijk} H L_k\,. \tag{11}$$

Hint: Prove and use the relation $(\mathbf{L}\wedge\mathbf{P})_j = -(\mathbf{P}\wedge\mathbf{L})_j + 2i\hbar P_j$. Relations derived in a) and given in the previous hint may also be useful.

Exercise 3: Hydrogen Atom – An algebraic approach (Oral)

We now bring in the harvest of the (technical) previous task with a derivation of the Hydrogen atom energy levels (i.e., the spectrum of H) by purely algebraic means. Historically, this was first achieved by Wolfgang Pauli even *before* Erwin Schrödinger published his famous equation.

a) Let $\mathbf{N} := \mathbf{M}/\sqrt{\lambda}$ with $\lambda \in \mathbb{R}_+$. Determine λ such that the following algebra is satisfied:

$$[L_i, L_j] = i\hbar\,\varepsilon_{ijk} L_k \tag{12a}$$

$$[L_i, N_j] = i\hbar\,\varepsilon_{ijk} N_k \tag{12b}$$

$$[N_i, N_j] = i\hbar\,\varepsilon_{ijk} L_k \tag{12c}$$

Hint: Use Eq. (11) and that H commutes with all operators N_i and L_i and therefore can be replaced by its eigenvalue.

b) Define the new operators

$$\mathbf{A} := \frac{1}{2}(\mathbf{L}+\mathbf{N}) \qquad \text{and} \qquad \mathbf{B} := \frac{1}{2}(\mathbf{L}-\mathbf{N}) \tag{13}$$

and derive their commutator algebra.

c) Derive the eigenvalues[2] of $\mathbf{A}^2$ and $\mathbf{B}^2$ by comparison with the known angular momentum algebra.

d) Define the new operators[3]

$$C_1 := \mathbf{A}^2 + \mathbf{B}^2 \qquad \text{and} \qquad C_2 := \mathbf{A}^2 - \mathbf{B}^2 \tag{14}$$

and show that $C_1 = \frac{1}{2}(\mathbf{L}^2 + \mathbf{N}^2)$ and $C_2 = 0$. Conclude that the eigenvalues of $\mathbf{A}^2$ and $\mathbf{B}^2$ coincide.

e) Use Eq. (9) to derive a relation between H and $\mathbf{L}^2$, $\mathbf{N}^2$ and solve it for H. Finally, derive the eigenvalues of H from the eigenvalues of $\mathbf{L}^2 + \mathbf{N}^2$ and compare them with the results from the lecture. Be happy.

[2]... for irreducible representations.

[3]In the theory of Lie algebras, those are known as *Casimir operators*. The Casimir operator of the angular momentum algebra is $\mathbf{L}^2$ and characterized by $\left[L_i, \mathbf{L}^2\right] = 0$ for all i.

Our christmas present for you:

There are no exercises on this page!