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Exercise 1: Density matrix (Written, 4 points)

a) A beam of spin-1 particles is incident on the modified Stern-Gerlach devices (α) and (β). The spin state of the beam is given by the density matrix (or *density operator*) $\rho_0 = \frac{1}{3}\mathbb{1}$. What are the density matrices ρ_α and ρ_β describing the beam after it has passed through (α) and (β), respectively ? Show that the density matrices obey $\mathrm{Tr}\,\rho_{\alpha/\beta} = 1$ and $\mathrm{Tr}\,\rho^2_{\alpha/\beta} \leq 1$. What is the difference between the situation with $\mathrm{Tr}\,\rho^2_{\alpha/\beta} = 1$ and that with $\mathrm{Tr}\,\rho^2_{\alpha/\beta} < 1$?

Figure 1: Stern-Gerlach devices (α) and (β).

Hint: Do not forget to normalize the density matrices !

b) We consider a beam of spin-1/2 particles whose state is described by the density operator

$$\rho_0 = |+\rangle \frac{1}{4} \langle +| + |-\rangle \frac{3}{4} \langle -|. \tag{1}$$

This beam is incident upon the Stern-Gerlach device (γ). What is the density operator ρ_γ of the beam after it has passed through the apparatus? Is there a pure state after the apparatus ?

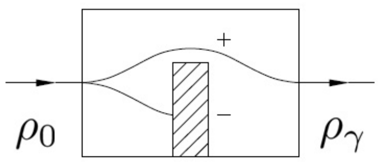

Figure 2: Stern-Gerlach device (γ).

c) Construct a density matrix for a beam of spin-1/2 particles that is fully polarized in the positive x-direction in the basis $\{|s_z, +\rangle, |s_z, -\rangle\}$. What is the expectation value for the polarization of the spin in y-direction ?

d) Construct a density matrix for a beam of spin-1/2 particles which consists of a fraction of 3/4 polarized in the positive z-direction and fraction of 1/4 polarized in the positive x-direction in the basis $\{|s_z, +\rangle, |s_z, -\rangle\}$. Compute the expectation values for s_x, s_y and s_z.

Exercise 2: Two-level system (Oral)

Consider the Hamiltonian

$$H = \hbar\omega \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2}$$

written in the basis $\{|a_1\rangle, |a_2\rangle\}$ where the states are labelled by the (non-degenerate) eigenvalues of an operator A.

Assume that at time $t = 0$ there is a statistical mixture described by the density matrix

$$\rho_0 = \frac{1}{4} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}. \tag{3}$$

How large is the probability $w_{a_1}(t)$ to obtain the measurement result a_1 when measuring A at a later time t.

Exercise 3: Harmonic oscillator in the Heisenberg picture (Written, 5 points)

The operators in the Heisenberg picture and operators in the Schrödinger picture are related through

$$A_H(t) = U^{-1}(t) A_S U(t) \quad \text{with} \quad U(t) = e^{-\frac{i}{\hbar}Ht}. \tag{4}$$

$U(t)$ is the time evolution operator which obeys the Schrödinger equation $i\hbar\partial_t U(t) = HU(t)$. We define the time-independent states in the Heisenberg picture as $|\psi_H\rangle := |\psi_S(t = 0)\rangle$. The index H stands for operators and states in the *Heisenberg* picture while the index S refers to the *Schrödinger* picture.

a) Derive the Heisenberg equation of motion for operators,

$$\partial_t A_H(t) = \frac{1}{i\hbar}[A_H(t), H], \tag{5}$$

starting from Eq. (4).

b) We now consider the Hamiltonian for a harmonic oscillator

$$H = \frac{P^2}{2m} + \frac{m\omega^2}{2} Q^2 = \hbar\omega \left(a_S^\dagger a_S + \frac{1}{2} \right), \quad [a_S, a_S^\dagger] = 1. \tag{6}$$

Show that

$$a_H(t) = e^{-i\omega t} a_S, \qquad a_H^\dagger(t) = e^{i\omega t} a_S^\dagger. \tag{7}$$

Hint: Let $a_H(t)$ act onto an eigenstate of the harmonic oscillator or use the Baker-Campbell-Hausdorff formula.

c) Use the results of the previous task to derive the relations

$$Q_H(t) = Q_H(t; P_S, Q_S), \qquad P_H(t) = P_H(t; P_S, Q_S). \tag{8}$$

Hint: $0 = e^{i\omega t} a_S - e^{i\omega t} a_S$ and $0 = e^{-i\omega t} a_S^\dagger - e^{-i\omega t} a_S^\dagger$.

d) Show for the harmonic oscillator that the Heisenberg equations of motion (5) lead to Hamilton's classical equations of motion for the operators $Q_H(t)$ und $P_H(t)$.

e) We define at time $t = 0$ the state $|\psi_S(t=0)\rangle = |\psi_H\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle) =: |\gamma\rangle$ (in the basis of the number operator $N|n\rangle = n|n\rangle$). Compute the expectation values

$$\langle Q_H(t)\rangle_\gamma = \langle\gamma|Q_H(t)|\gamma\rangle, \qquad \langle P_H(t)\rangle_\gamma = \langle\gamma|P_H(t)|\gamma\rangle, \tag{9}$$

for arbitrary times t.