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Exercise 1: Spin-orbit coupling and Zeeman effect

We consider the Hamiltonian H_0 of a hydrogen atom which is perturbed by

$$H_{SO} = \frac{1}{2m_e^2c^2r}\partial_r V(r)\boldsymbol{L}\cdot\boldsymbol{S} \tag{1}$$

and

$$H_B = \frac{e}{2m}\boldsymbol{B}\cdot(\boldsymbol{L}+2\boldsymbol{S})\,, \tag{2}$$

where $\boldsymbol{L}$ is the orbital angular momentum and $\boldsymbol{S}$ is the spin of the electron. $V(r)$ is the electric potential of the core. The first Hamiltonian, (1), describes the spin-orbit coupling while the second one, (2), describes the coupling to an external magnetic field $\boldsymbol{B}$ which we assume to be constant and pointing along the z-axis.

a) Show that both perturbations commute with L^2 and that H_{SO} does not commute with H_0. Explain how first order perturbation theory is used.

b) Discuss (calculate) the partial splitting of the $2p$-level under the perturbation H_{SO}.

c) Consider a weak magnetic field such that $\langle H_B\rangle \ll \langle H_{SO}\rangle$ and we can treat (2) as a perturbation to $H = H_0 + H_{SO}$. Calculate the the splitting of the energy levels calculated in b).

d) Consider now the case of a strong magnetic field with $\langle H_B\rangle \gg \langle H_{SO}\rangle$, that is, treat (1) as a perturbation to $H = H_0 + H_B$. In the limit of vanishing H_{SO}, this is called the *Zeeman effect*. Calculate the splitting of the $2p$-level for this case and draw schematically the evolution of the energy levels when going from a weak to a strong magnetic field.

e) Calculate exactly the evolution of the energy levels in the $2p$ subspace by using the dimensionless Hamiltonian

$$H_I(\lambda_B) = \frac{1}{\hbar^2}\boldsymbol{L}\cdot\boldsymbol{S} + \frac{\lambda_B}{\hbar}(\boldsymbol{L}_z + 2\boldsymbol{S}_z) \tag{3}$$

and plot the energy levels as a function of λ_B.
Hint: You can also use Maple or Mathematica.

Exercise 2: Interacting spin -1/2

Consider a system composed of three spin-1/2 particles.

a) What is the dimension of the Hilbert space? Define the total spin operator as $\mathbf{S} = \sum_{i=1}^{3} \mathbf{S}^{(i)}$ and its z projection as $S_z = \sum_{i=1}^{3} S_z^{(i)}$ and write down a complete basis of the Hilbert space.

b) The Hamiltonian of the system is

$$H = J \sum_{i=1}^{3} \mathbf{S}^{(i)} \cdot \mathbf{S}^{(i+1)}, \quad J > 0.$$

Here we assume a periodic system (for $i = 3$ take $i + 1 = 1$). Calculate the eigenstates and eigenenergies of this Hamiltonian.

Hint:: Rewrite H as a function of S^2 and $S^{(i)2}$.