Exercise 1: Spin-orbit coupling and Zeeman effect

We consider the Hamiltonian $H_0$ of a hydrogen atom which is perturbed by

$$H_{SO} = \frac{1}{2m_e^2c^2r} \partial_r V(r) \mathbf{L} \cdot \mathbf{S}$$

and

$$H_B = \frac{e}{2m} \mathbf{B} \cdot (\mathbf{L} + 2\mathbf{S}) ,$$

where $\mathbf{L}$ is the orbital angular momentum and $\mathbf{S}$ is the spin of the electron. $V(r)$ is the electric potential of the core. The first Hamiltonian, (1), describes the spin-orbit coupling while the second one, (2), describes the coupling to an external magnetic field $\mathbf{B}$ which we assume to be constant and pointing along the $z$-axis.

a) Show that both perturbations commute with $L^2$ and that $H_{SO}$ does not commute with $H_0$. Explain how first order perturbation theory is used.

b) Discuss (calculate) the partial splitting of the $2p$-level under the perturbation $H_{SO}$.

c) Consider a weak magnetic field such that $\langle H_B \rangle \ll \langle H_{SO} \rangle$ and we can treat (2) as a perturbation to $H = H_0 + H_{SO}$. Calculate the the splitting of the energy levels calculated in b).

d) Consider now the case of a strong magnetic field with $\langle H_B \rangle \gg \langle H_{SO} \rangle$, that is, treat (1) as a perturbation to $H = H_0 + H_B$. In the limit of vanishing $H_{SO}$, this is called the Zeeman effect. Calculate the splitting of the $2p$-level for this case and draw schematically the evolution of the energy levels when going from a weak to a strong magnetic field.

e) Calculate exactly the evolution of the energy levels in the $2p$ subspace by using the dimensionless Hamiltonian

$$H_I(\lambda_B) = \frac{1}{\hbar^2} \mathbf{L} \cdot \mathbf{S} + \frac{\lambda_B}{\hbar} (L_z + 2S_z)$$

and plot the energy levels as a function of $\lambda_B$.

*Hint:* You can also use Maple or Mathematica.
Exercise 2: Interacting spin -1/2

Consider a system composed of three spin-1/2 particles.

a) What is the dimension of the Hilbert space? Define the total spin operator as $S = \sum_{i=1}^{3} S^{(i)}$ and its $z$ projection as $S_z = \sum_{i=1}^{3} S_z^{(i)}$ and write down a complete basis of the Hilbert space.

b) The Hamiltonian of the system is

$$ H = J \sum_{i=1}^{3} S^{(i)} \cdot S^{(i+1)}, \quad J > 0. $$

Here we assume a periodic system (for $i = 3$ take $i + 1 = 1$). Calculate the eigenstates and eigenenergies of this Hamiltonian.

*Hint*: Rewrite $H$ as a function of $S^2$ and $S^{(i)2}$. 