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You can find detailed information about the lecture and the exercises on the website `http://www.itp3.uni-stuttgart.de/lehre/vorlesungen/QM2.html`. Exercises are divided into two different groups. *Written* exercises have to be handed in and will be graded by the tutors. *Oral* exercises have to be prepared for the exercise session and will be presented by one of the students. In order to be admitted to the exam, we require (a) 80% of the written exercises to be solved or treated adequately, (b) 66% of the oral exercises to be prepared and (c) two exercises to be presented at the black board.

The first exercise sheet serves as a repetition for some important concepts from your first quantum mechanics lecture.

Exercise 1: Foundations of quantum mechanics (Oral)

This exercise is based on *Student understanding of quantum mechanics*, C. Singh, Am. J. Phys. **69**, 885 (2001); `http://dx.doi.org/10.1119/1.1365404`.

Try to take the following short test without using any reference material.

We refer to a generic observable Q and its corresponding quantum mechanical operator $\hat{Q}$. For all of the questions, the Hamiltonian and operators $\hat{Q}$ do not depend upon time explicitly.

a) The eigenvalue equation for an operator $\hat{Q}$ is given by $\hat{Q}\,|\psi_i\rangle = \lambda_i\,|\psi_i\rangle$, where $i = 1, \ldots, N$. Write an expression for $\langle\phi|\hat{Q}|\phi\rangle$, where $|\phi\rangle$ is a general state, in terms of the projections $\langle\phi|\psi_i\rangle$.

b) If you make measurements of a physical observable Q on a system in immediate succession, do you expect the outcome to be the same every time? Justify your answer.

c) If you make measurements of a physical observable Q on an ensemble of identically prepared systems which are not in an eigenstate of $\hat{Q}$, do you expect the outcome to be the same every time? Justify your answer.

d) A particle is in a one-dimensional harmonic oscillator potential. Under what conditions will the expectation value of an operator $\hat{Q}$ depend on time if

   i. the particle is initially in a momentum eigenstate?

   ii. the particle is initially in an energy eigenstate?

   Justify your answer in both cases.

e) Questions (i)-(ix) refer to the following system. An electron is in a uniform magnetic field B which is pointing in the z direction. The Hamiltonian for the spin-degree of freedom for this system is given by $\hat{H} = -\gamma B \hat{S}_z$, where γ is the gyromagnetic ratio and $\hat{S}_z$ is the z component of the spin angular momentum operator. Notation:

$\hat{S}_z\,|\uparrow\rangle = \hbar/2\,|\uparrow\rangle\,, \hat{S}_z\,|\downarrow\rangle = -\hbar/2\,|\downarrow\rangle$.
For reference, the unnormalized eigenstates of $\hat{S}_x$ and $\hat{S}_y$ are given by

$$\begin{aligned}\hat{S}_x(|\uparrow\rangle \pm |\downarrow\rangle) &= \pm\hbar/2(|\uparrow\rangle \pm |\downarrow\rangle)\\ \hat{S}_y(|\uparrow\rangle \pm i\,|\downarrow\rangle) &= \pm\hbar/2(|\uparrow\rangle \pm i\,|\downarrow\rangle)\end{aligned} \tag{1}$$

i. If you measure S_z in a state $|\chi\rangle = (|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$, what are the possible results, and what are their respective probabilities?

ii. If the result of the first measurement of S_z was $\hbar/2$, and you immediately measure S_z again, what are the possible results, and what are their respective probabilities?

iii. If the result of the first measurement of S_z was $\hbar/2$, and you immediately measure S_x, what are the possible results, and what are their respective probabilities?

iv. What is the expectation value $\langle\hat{S}_z\rangle$ in the state $|\chi\rangle = (|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$?

v. If the electron is initially in an eigenstate of $\hat{S}_x$, does the expectation value of $\hat{S}_y$ depend on time? Justify your answer.

vi. If the electron is initially in an eigenstate of $\hat{S}_x$, does the expectation value of $\hat{S}_z$ depend on time? Justify your answer.

vii. If the electron is initially in an eigenstate of $\hat{S}_z$, does the expectation value of $\hat{S}_x$ depend on time? Justify your answer.

viii. If the electron is initially in an eigenstate of $\hat{S}_z$, does the expectation value of $\hat{S}_z$ depend on time? Justify your answer.

ix. If the electron is initially in an eigenstate of $\hat{S}_x$, does the expectation value of $\hat{S}_x$ depend on time? Justify your answer.

Exercise 2: Commutators (Written, 5p)

a) Show that the following identity holds for commutators of products:

$$[A, BC] = B[A, C] + [A, B]C \tag{2}$$

b) Let $g(x)$ and $f(p)$ be analytical functions of the position and momentum operators, respectively. Show that

$$[p, g(x)] = -i\hbar \frac{\mathrm{d}}{\mathrm{d}x} g(x)$$
$$[x, f(p)] = +i\hbar \frac{\mathrm{d}}{\mathrm{d}p} f(p) \tag{3}$$

c) Calculate the following commutators for canonical position and momentum operators x and p:

$$\left[x, p^2\right], \qquad \left[x^2, p^2\right], \qquad \left[xp, p^2\right] \tag{4}$$

d) Using just the canonical commutation relations of x and p, calculate $[L_\alpha, L_\beta]$, where $L_\alpha = \epsilon_{\alpha ij} x_i p_j$ is the α-component of the angular momentum operator.

e) Now calculate $[L_\alpha, \mathbf{L}^2]$ and $[L_\alpha, p^2]$.

Exercise 3: Creation and annihilation operators (Oral)

The Hamiltonian of a one-dimensional harmonic oscillator is given by

$$H = \frac{p^2}{2m} + \frac{m}{2}\omega^2 x^2. \tag{5}$$

a) As usual, we introduce creation and annihilation operators $a^\dagger$ and a via

$$a = \sqrt{\frac{m\omega}{2\hbar}} \left(x + \frac{i}{m\omega} p\right), \quad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left(x - \frac{i}{m\omega} p\right). \tag{6}$$

Calculate the commutation relations for these operators.

b) Express the Hamiltonian in terms of the number operator $n = a^\dagger a$.

c) Let $|n\rangle$ be the eigenvector of $a^\dagger a$ with eigenvalue n. How can $|n\rangle$ be expressed in terms of the creation operator $a^\dagger$ and the ground state $|0\rangle$ (no calculation needed)?

d) What is the ground state wavefunction $\psi_0(x) = \langle x|0\rangle$? Use the coordinate representation of the creation operator to calculate the excited state $\psi_1(x) = \langle x|1\rangle$.

e) Show that the creation operator $a^\dagger$ has no (right-)eigenvector. Does the same argument hold for the annihilation operator a?