Prof. Dr. Hans Peter Büchler
Institut für Theoretische Physik III, Universität Stuttgart

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The second exercise sheet will consider a repetition of time-independent perturbation theory, as well as the time-dependent case.

Exercise 1: Stark effect for a harmonic oscillator (Oral)

The Hamiltonian of a one-dimensional harmonic oscillator within a homogeneous electric field E is given by

$$\mathrm{H} = \frac{1}{2}\left(P^2 + Q^2\right) + eEQ\,, \tag{1}$$

where the Hamiltonian is dimensionless. Consider the second term of the Hamiltonian as a perturbation to the free oscillator scenario, such that $\mathrm{H}_1 = Q$, with $\lambda = eE$.

a) Calculate the perturbed eigenfunctions and energy eigenvalues up to third order in λ.

b) Compare the result of perturbation theory with the exact solution for the energy of the problem.

Exercise 2: Fermi's Golden Rule (Written, 4 pts)

Derive Fermi's Golden Rule for the periodic perturbation

$$\mathrm{H}_1 = V\ \cos(\omega t)\exp(\eta t)\ , \tag{2}$$

where $\eta > 0$.

Exercise 3: Time-dependent perturbation theory (Oral)

We investigate a one-dimensional harmonic oscillator with mass m and frequency ω within a time-dependent electric field. The Hamiltonian is of the form

$$\begin{aligned} &\mathrm{H} = \mathrm{H}_0 + \mathrm{H}'(t)\ , \\ &\text{where} \quad \mathrm{H}_0 = \frac{p^2}{2m} + \frac{m}{2}\omega^2 x^2 \quad \text{(harmonic oscillator)} \\ &\text{and} \quad \mathrm{H}'(t) = exE(t) \quad \text{(perturbation)}\,. \end{aligned} \tag{3}$$

The time dependency of the external electric field is given as

$$E(t) = \frac{A}{\tau\sqrt{\pi}}\, e^{-(t/\tau)^2}\ , \tag{4}$$

where A is a constant and $\tau > 0$ is the decay rate.

a) Calculate the transition probability from the ground state to an excited state $P_{0\to n}(\infty)$ in first order of perturbation theory. What happens for $\tau \to 0$?

b) The transition probability can also be calculated exactly. Prove that the exact expression for the transition probability is

$$P_{0\to n}(\infty) = \frac{K^{2n}}{n!} \exp\left(-K^2\right) \quad \text{with} \quad K = \frac{eA\, \exp\left(-\omega^2\tau^2/4\right)}{\sqrt{2m\omega\hbar}}\ , \tag{5}$$

and compare the result with task a).
Hint: Use the Dirac picture, as introduced in the lecture, and derive the equation of motion

$$i\hbar\partial_t|\psi_D(t)\rangle = \mathrm{H}_D|\psi_D(t)\rangle \quad \text{with} \quad \mathrm{H}_D = \sqrt{\frac{\hbar}{2m\omega}}\, D(t)\, \left[e^{-i\omega t}a + e^{i\omega t}a^\dagger\right]\,. \tag{6}$$

Use the ansatz $|\psi_D(t)\rangle = \exp\left(-iK(t)\, a^\dagger\right)|\bar{\psi}(t)\rangle$ and choose $K(t)$ as

$$K(t, t_i) = \frac{e}{\sqrt{2m\hbar\omega}} \int_{t_i}^{t} e^{i\omega t'}\, E(t')\, \mathrm{d}t'\ , \tag{7}$$

such that $a^\dagger$ can be eliminated from the Schrödinger equation. The relation $e^{iKa^\dagger} a e^{-iKa^\dagger} = a - iK$ then helps solving the remaining equation and hence gives the exact solution $|\psi(t)\rangle$ of the wave function within the Schrödinger picture. The transition probability then is $P_{0\to n}(\infty) = |\langle n|\psi(\infty)\rangle|^2$.