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The third exercise sheet will deal with adding angular momenta.

Exercise 1: Clebsch-Gordon coefficients and spin-orbit coupling (Oral)

Relativistic description of the hydrogen atom via Dirac equation provides corrections to the nonrelativistic Hamiltonian H_0. One of the important terms is the so-called spin-orbit coupling term, which can be intuitively understood as the interaction of the electron spin $\mathbf{S}$ with the magnetic field generated by its own orbital motion. The relativistic calculations for hydrogen atom give the following formula describing this effect:

$$H_{\mathrm{LS}} = f(r)\mathbf{L}\cdot\mathbf{S} = f(r)\sum_{i=x,y,z} L_i \otimes S_i, \tag{1}$$

where $f(r) = e^2/2m_e^2c^2r^3 4\pi\epsilon_0$. We can regard this term as a perturbation to the nonrelativistic Hamiltonian H_0.

a) Show that if we define the total angular momentum operator as $\mathbf{J} = \mathbf{L} + \mathbf{S} = \mathbf{L}\otimes \mathbb{1} + \mathbb{1}\otimes\mathbf{S}$, $\mathbf{J}^2$ commutes with H_0 and H_{LS}.

b) For the subspace of states with $\ell = 1$ (the spin $s = 1/2$), derive the eigenstates of $\mathbf{J}^2$ and J_z (denoted as $|j,m\rangle$) as linear combinations of $|m_\ell, m_s\rangle = |\ell = 1, m_\ell\rangle \otimes |s = 1/2, m_s\rangle$ states

$$|j,m\rangle = \sum_{m_\ell, m_s} c(m_\ell, m_s; j, m)\, |m_\ell, m_s\rangle. \tag{2}$$

The coefficients c are called Clebsch-Gordon coefficients.

Tip: Start with the $j = 3/2$, $m_j = 3/2$ state and use ladder operator $J_- = J_x - iJ_y$, which acts according to the formula

$$J_- \,|j,m\rangle = \hbar\sqrt{j(j+1) - m(m-1)}\,|j, m-1\rangle\,.$$

Exercise 2: Fine structure of the hydrogen atom (Written, 4p)

Calculate the energy corrections arising from the spin-orbit coupling to the $2p$ levels of the hydrogen atom (with the principal quantum number $n = 2$ and orbital angular momentum $\ell = 1$) within the first order of perturbation theory.

Tip: For the radial part $\langle\psi_{2,1,m}|1/r^3|\psi_{2,1,m}\rangle = 1/24a_0^3$, where $\psi_{2,1,m}$ is the eigenfunction of H_0, and $a_0 = 4\pi\epsilon_0\hbar^2/m_e e^2$ is the Bohr radius.

Exercise 3: Interacting 1/2 spins (Oral)

Let us consider a system composed of three spin-1/2 particles.

a) What is the dimension of the Hilbert space?

The total spin operator can be defined as $\mathbf{S} = \sum_{i=1}^{3} \mathbf{S}^{(i)}$ and its z projection as $S_z = \sum_{i=1}^{3} S_z^{(i)}$. What are the eigenvalues and eigenstates of $\mathbf{S}^2$ and S_z?

b) The Hamiltonian of the system is

$$H = J \sum_{i=1}^{3} \mathbf{S}^{(i)} \cdot \mathbf{S}^{(i+1)}, \quad J > 0.$$

Here we assume a periodic system (for $i = 3$ take $i + 1 = 1$). Calculate the eigenstates and eigenenergies of this Hamiltonian.

Tip: Rewrite H as a function of S^2 and $S^{(i)2}$.