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Exercise 1: Identical fermions in a potential well (Written, 4pts)

We consider two identical spin-1/2 fermions in a one-dimensional potential given by

$$V(x) = \begin{cases} 0 & |x| \leq 1 \\ \infty & \text{otherwise}\,. \end{cases} \tag{1}$$

The (dimensionless) single-particle Hamiltonian for the i-th particle is given by

$$H^{(i)} = -\frac{1}{2}\partial_{x_i}^2 + V(x_i). \tag{2}$$

a) Explain why we can treat the orbital motion and the spin dynamics separately, i.e. explain why we can write the single-particle states as a product of orbital- and spin components.

   Specify the two single-particle wave functions (orbital part) that are lowest in energy.

b) Determine the ground state of a two-fermion system with $H = \sum_{i=1}^{2} H^{(i)}$ for the following two cases:

   i. For a spin state that is antisymmetric under exchange of the two fermions, i.e. the singlet spin state $(|{\uparrow\downarrow}\rangle - |{\downarrow\uparrow}\rangle)/\sqrt{2}$.

   ii. For a spin state that is symmetric under exchange of the two fermions, i.e. one of the triplet spin states $|{\uparrow\uparrow}\rangle$, $|{\downarrow\downarrow}\rangle$ or $(|{\uparrow\downarrow}\rangle + |{\downarrow\uparrow}\rangle)/\sqrt{2}$.

c) Examine the influence of a contact-interaction between the two fermions, which is described by the interaction potential $\lambda\delta(x_1 - x_2)$ with strength $\lambda \in \mathbb{R}$. To this end, calculate the energy correction in first order perturbation theory ($|\lambda| \ll 1$) for both the singlet states and triplet states.
   Give an explanation for why the perturbation theory result for the triplet-state is correct for all λ.

Exercise 2: Ideal Fermi gas in one dimension (Oral)

Consider an ideal (non-interacting) gas of N fermions in a one-dimensional volume of length L with periodic boundary conditions and Hamiltonian

$$H = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2}\,. \tag{3}$$

The single-particle wave functions are given by $\phi_{k,\sigma}(x,s) = L^{-1/2}\exp(ikx)\,\chi_\sigma(s)$ where k is the (one-dimensional) wave vector, x is the spacial coordinate, σ is the spin and s the spin coordinate. The periodic boundary conditions impose $\phi_{k,\sigma}(x+L) = \phi_{k,\sigma}(x)$.

a) Give an expression for the ground state wave function of the general N-particle problem (even N). Calculate the wave function for the two-particle problem explicitly.

b) Calculate the dispersion relation $\epsilon(k)$. Use the result to determine the ground state energy for N particles. What is the Fermi energy? (Hint: you might want to assume that the number of particles can be written as $N = 2 + 4n$ where $n \in \mathbb{N}$)

c) Consider the case $L \longrightarrow \infty$. In this limit, we can write the density of states in terms of an integral:

$$\rho(E) = \frac{1}{L}\sum_{k,\sigma}\delta(E-\epsilon(k)) = \frac{2}{L}\sum_{k}\delta(E-\epsilon(k)) \longrightarrow 2\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\,\delta(E-\epsilon(k)). \tag{4}$$

Evaluate this integral. Note that the density of states has units of $(\text{length}\cdot\text{energy})^{-1}$, i.e. $(\text{volume}\cdot\text{energy})^{-1}$.

d) Calculate the energy density using the density of states.