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Exercise 1: Commutator of the electric field (Oral)

In this problem we calculate the commutator $\left[\hat{E}_i(\mathbf{x},t)\ ,\ \hat{E}_j(\mathbf{x}',t')\right]$ of the electric field.

a) Calculate the commutator $\left[\hat{A}_i(\mathbf{x},t)\ ,\ \hat{A}_j(\mathbf{x}',t')\right]$ as the first step.
Begin your computations by decomposing the vector potential $\mathbf{A}(\mathbf{x},t)$ into its modes

$$\mathbf{A}(\mathbf{x},t) = \sum_{\mathbf{k},\lambda} \left(\frac{hc^2}{V\omega_{\mathbf{k}}}\right)^{1/2} \left(\hat{a}_{\mathbf{k},\lambda}\boldsymbol{\epsilon}(\mathbf{k},\lambda)\ e^{i(\mathbf{k}\cdot\mathbf{x}-\omega_{\mathbf{k}}t)} + \text{h.c.}\right)\ , \tag{1}$$

where $\mathbf{x}$ is the spatial coordinate, t the time, $\mathbf{k}$ the wave vector, V is the considered volume, h is the Planck's constant, c is the speed of light, $\omega_{\mathbf{k}} = c|\mathbf{k}|$, $\hat{a}_{\mathbf{k},\lambda}$ the annihilation operator for wave number $\mathbf{k}$ and polarization λ and $\boldsymbol{\epsilon}$ is the vector of polarization, see the lecture notes by Blatter 2005 p. 458. In addition, use the relation of completeness

$$\sum_{\lambda} \epsilon_i(\mathbf{k},\lambda)\ \epsilon_j^*(\mathbf{k},\lambda) = \delta_{ij} - \frac{k_i k_j}{k^2}\ , \tag{2}$$

of the vectors of polarization in order to bring the commutator into the following form

$$\left[\hat{A}_i(\mathbf{x},t)\ ,\ \hat{A}_j(\mathbf{x}',t')\right] = \partial_{ij}\, K(\boldsymbol{\xi},\tau)\ , \tag{3}$$

with $\boldsymbol{\xi} \equiv \mathbf{x}-\mathbf{x}'$ and $\tau \equiv t-t'$, where $K(\boldsymbol{\xi},\tau)$ has to be determined. The operator for the derivation ∂_{ij} is defined as

$$\partial_{ij} \equiv \frac{\partial_\tau^2}{c^2}\,\delta_{ij} - \partial_{\xi_i}\,\partial_{\xi_j}\ . \tag{4}$$

b) Write the commutator for the electric field $\mathbf{E}$ in the following form

$$\left[\hat{E}_i(\mathbf{x},t)\ ,\ \hat{E}_j(\mathbf{x}',t')\right] = -\frac{1}{c^2}\frac{\partial^2}{\partial_\tau^2}\left[\hat{A}_i(\mathbf{x},t)\ ,\ \hat{A}_j(\mathbf{x}',t')\right]\ , \tag{5}$$

c) Next, transform the commutator into a form, which is proportional to $\delta(\xi^2 - c^2\,\tau^2)$.
Give and explain the physical concept behind this solution.

Hint: It is not required to evaluate the derivatives ∂_{ij}, it is sufficient to perform the integration over $\mathbf{k}$. This can be done by replacing the summation $\frac{1}{V}\sum_{\mathbf{k}} \to \int \frac{\mathrm{d}^3k}{(2\pi)^3}$ by an integral.

d) What happens if the photons would follow fermionic statistics? Demand the involved operators to anti-commute $\left\{\hat{a}_{\mathbf{k},\lambda}\,,\,\hat{a}^{\dagger}_{\mathbf{k}',\lambda'}\right\} = \delta_{\lambda,\lambda'}\,\delta(\mathbf{k}-\mathbf{k}')$. Calculate the anti-commutator of the electric field $\left\{\hat{E}_i(\mathbf{x},t)\;,\;\hat{E}_j(\mathbf{x}',t')\right\}$.

Exercise 2: Coherent states (Written, 5 points)

A coherent (or Glauber-) state $|\alpha\rangle$ is defined as the right eigenstate of the annihilation operator

$$\hat{\alpha}\,|\alpha\rangle = \alpha\,|\alpha\rangle\,, \tag{6}$$

with eigenvalue $\alpha \in \mathbb{C}$.

a) Determine the coefficients $c_n(\alpha)$ of the expansion $|\alpha\rangle = \sum_{n=0}^{\infty} c_n(\alpha)\,|n\rangle$ for the normalized coherent state $\langle\alpha|\alpha\rangle = 1$, where $|n\rangle$ is the eigenstate of the occupation operator $\hat{n}$.

b) Find the operator $\hat{C}(\alpha)$, which creates a coherent state $|\alpha\rangle = \hat{C}(\alpha)\,|0\rangle$, when applied to the ground state $|0\rangle$.

c) Calculate the mean particle number $\langle\alpha|\hat{n}|\alpha\rangle$ and the quadratic deviation $(\Delta\hat{n})^2$. What kind of probability distribution does this relate to?

d) Consider a one-dimensional oscillator with the Hamiltonian $\hat{\mathrm{H}} = \hbar\omega\left(\hat{n}+\frac{1}{2}\right)$, which is in the initial state $|\psi(t=0)\rangle = |\alpha\rangle$. Demonstrate, that the time evolution can be written as $|\psi(t)\rangle \sim |\alpha(t)\rangle$ (up to a phase) with a time-dependent $\alpha(t)$.

e) Compute the product $\langle\alpha|\alpha'\rangle$, as well as $\int \mathrm{d}^2\alpha\;|\alpha\rangle\langle\alpha|$. Use polar coordinates for the integration within the complex plane. How can the solution be interpreted?