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Exercise 1: The Casimir effect (Written, volunteer, 14 points)

Owing to quantum fluctuations of the electromagnetic field, there is an attractive force between two parallel metallic plates separated by a distance d, even if the two plates are located in a vacuum and are electrically neutral. This is known as the *Casimir effect*. As we will see in this exercise, for two plates of area A separated by a distance d, the energy shift due to vacuum fluctuations is

$$U(d,A) = -\frac{\pi^2}{720}\frac{\hbar A}{d^3}. \tag{1}$$

Due to this energy shift, the force between the two plates is non zero and attractive

$$F = -\frac{\partial U(d,A)}{\partial d} = -\frac{\pi^2}{240}\frac{\hbar A}{d^4}\,. \tag{2}$$

This has been confirmed experimentally in 1958 by Sparnay (It was realized using $1cm^2$ Chrome-Steal plates; at $d = 0.5\mu$ the attraction was $0.2dyn/cm^2$).

a) Let us consider an electromagnetic field confined in a rectangular cavity (of dimensions $L_1 \times L_2 \times L_3$) with conducting walls. We must have $\mathbf{E}$ perpendicular and $\mathbf{B}$ tangential (the transverse component of the electric field vanishes at the surface of a perfect conductor). Show that these boundary conditions are satisfied by plane waves ($\sim e^{-i\omega t}$) if the components of the electric field have the following form

$$E_1 = E_1^0 \cos(k_1 x_1)\,\sin(k_2 x_2)\,\sin(k_3 x_3)\,e^{-i\omega t} \tag{3}$$
$$E_2 = E_2^0 \sin(k_1 x_1)\,\cos(k_2 x_2)\,\sin(k_3 x_3)\,e^{-i\omega t} \tag{4}$$
$$E_3 = E_3^0 \sin(k_1 x_1)\,\sin(k_2 x_2)\,\cos(k_3 x_3)\,e^{-i\omega t}\,, \tag{5}$$

where $k_i = n_i\pi/L_i$ and $n_i \in \mathbb{Z}$ and that the possible frequencies ω are restricted by the dispersion relation of light

$$\frac{1}{c^2}\omega^2(n_1,n_2,n_3) = \mathbf{k}^2 = \pi^2\sum_i (n_i^2/L_i^2). \tag{6}$$

b) Show that the corresponding boundary conditions for the magnetic field $\mathbf{B}$ are fulfilled automatically. Recall that the magnetic field $\mathbf{B}$ is related to the electric field by the induction law $\nabla \times \mathbf{E} = i\,(\omega/c)\,\mathbf{B}$.

c) The amplitudes E_i^0 are fixed by the condition $\nabla \cdot \mathbf{E} = 0$, *i.e.* and thus satisfy

$$\sum_i E_i^0 k_i = 0. \tag{7}$$

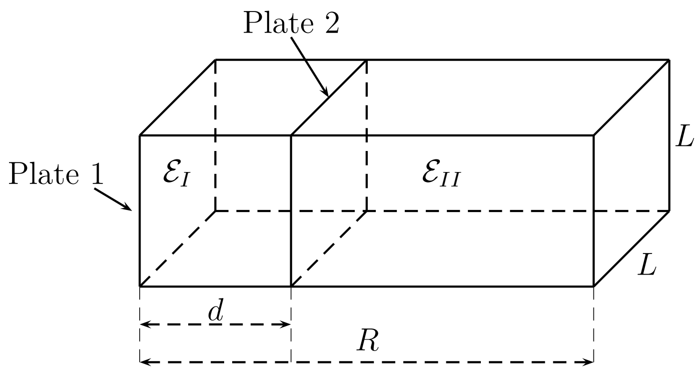

Figure 1: Setup of the two plates used to measure the Casimir effect.

Show that in general equation (7) has two linearly independent solutions, corresponding to the two polarizations of the electromagnetic field, except when one of the n_i vanish, that there is just one solution. If more than one vanish then there is no solution.

d) Consider now two conducting and non-charged plates of dimensions $L \times L$ placed in a parallelogram with conducting walls as shown in the Figure 1. One conducting plate is fixed at the beginning of the box, while the second plate is chosen to be at a distance d from the former. This second plate will be moved to a distance R/η (with arbitrary $\eta > 0$) in a forthcoming step. We can define

$$U(d, L, R) := \mathcal{E}_I(d) + \mathcal{E}_{II}(R - d) - [\mathcal{E}_{III}(R/\eta) + \mathcal{E}_{IV}(R - R/\eta)] \ , \tag{8}$$

as the energy difference between the zero point energies of the initial and final configurations, where $\mathcal{E}_I$, $\mathcal{E}_{II}$, $\mathcal{E}_{III}$, $\mathcal{E}_{IV}$ refer to the zero-point energy of each subspace, respectively. Show that each of them is divergent.

Defining these subspaces are indeed a tool to avoid divergences, as we are actually interested in taking the limit

$$U(d, L) = \lim_{R\to\infty} U(d, L, R) \ . \tag{9}$$

Thus, we need first to regularize the sums of the zero-point energy prior to calculation of Eq. (9). After the computation of Eq. (9), we will undo the regularization.

e) A convenient regularization method is the following

$$\mathcal{E}_{I,II} \to \mathcal{E}_{I,II}^{reg} = \sum_{\omega} \frac{1}{2} \hbar\omega \ \exp[-\alpha\omega/\pi c] \ . \tag{10}$$

Taking into account the dispersion relation (6) we have

$$\mathcal{E}_I^{reg} = \hbar c \sum_{l,m,n} k_{l,m,n}(d, L, L) \ \exp[-(\alpha/\pi)\, k_{l,m,n}(d, L, L))] \,, \tag{11}$$

where

$$k_{l,m,n}(d, L, L) = \sqrt{\left(\frac{l\pi}{d}\right)^2 + \left(\frac{m\pi}{L}\right)^2 + \left(\frac{n\pi}{L}\right)^2} \ . \tag{12}$$

We can define the regularized energy difference Eq. (8) as

$$U^{reg}\,(d,L,R,\alpha) = \mathcal{E}_I^{reg}(d) + \mathcal{E}_{II}^{reg}(R-d) - \{\mathcal{E}_{III}^{reg}(R/\eta) + \mathcal{E}_{IV}^{reg}(R-R/\eta)\}\;. \quad (13)$$

f) Let us consider the sum in equation (11). Consider very large L and replace the sums over m and n by integrals (a more precise way would be to study $U^{reg}(d,R,L^2,\alpha)/L^2$ when L goes to infinity) obtaining

$$\begin{aligned}\mathcal{E}_I^{reg}(d,L,\alpha) &= \hbar c \sum_{l=0}^{\infty} \int_0^{\infty} dm \int_0^{\infty} dn \sqrt{\left(\frac{l\pi}{d}\right)^2 + \left(\frac{m\pi}{L}\right)^2 + \left(\frac{n\pi}{L}\right)^2} \times \\ &\qquad \exp\left[-\frac{\alpha}{\pi}\sqrt{\left(\frac{l\pi}{d}\right)^2 + \left(\frac{m\pi}{L}\right)^2 + \left(\frac{n\pi}{L}\right)^2}\,\right]. \end{aligned} \quad (14)$$

In equation (13) the term with $l=0$ does not contribute to the sum. Therefore we can neglect it. Transform equation (14) into

$$\mathcal{E}_I^{reg} = -\frac{\pi^2}{4}\,\hbar c\, L^2 \,\frac{d^3}{d\alpha^3} \sum_{l=1}^{\infty} \int_0^{\infty} \frac{dz}{1+z} \exp\left[-\frac{l}{d}\,\alpha\,\sqrt{1+z}\,\right]. \quad (15)$$

Perform the sum over l and then take the derivative with respect to α, arriving to

$$\mathcal{E}_I^{reg} = \frac{\pi^2 \hbar c L^2}{2d}\,\frac{d^2}{d\alpha^2}\,\frac{d/\alpha}{\exp[\alpha/d]-1}\,. \quad (16)$$

Hint: $\frac{y}{e^y-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} y^n$ where the B_n are the Bernoulli numbers.

g) Calculate U^{reg} and obtain Eq. (1) by taking the limits

$$\lim_{R\to\infty}\,\lim_{\alpha\to 0} U^{reg}(d,L,R,\alpha) \quad (17)$$

Exercise 2: Quantum collapses and revivals (Oral)

In this exercise, we will study the effect of a coherent single-mode electromagnetic field interacting with a two-level atom in a cavity. We will see that the revival of the atomic population inversion after its collapse is a direct consequence of the quantum nature of the electromagnetic field.

a) Show that the Hamiltonian operator for a single mode electromagnetic field interacting resonantly with a two-level atom can be written as

$$H = H_0 + H_1, \quad H_0 = \hbar\omega(b^\dagger b + \sigma^z), \quad H_1 = \hbar g(b\sigma^+ + b^\dagger\sigma^-) \quad (18)$$

where the operators $b, b^\dagger$ correspond to the single mode of the electromagnetic field of the cavity, and $\sigma^z, \sigma^+, \sigma^-$ to the spin operator of the atom. This is known as the the Jaynes-Cummings model.

b) Show that $[H_0, H_1] = 0$ meaning that the Hamiltonian (18) is exactly solvable.

c) The eigenstates of H_0 can be labeled by the number of photons and the level of the atom, i.e., $|n, s\rangle$, where $n = 0, 1, 2\ldots$ and $s = 0, 1$. Show that the eigenstates of the complete Hamiltonian (18) are

$$|\phi_n^+\rangle = \frac{1}{\sqrt{2}}(|n,1\rangle + |n+1,0\rangle), \tag{19}$$

$$|\phi_n^-\rangle = \frac{1}{\sqrt{2}}(|n,1\rangle - |n+1,0\rangle) \tag{20}$$

with eigenvalues $\pm\Omega\hbar$, where $\Omega = g\sqrt{n+1}$.

d) Given that the atom is initially in the excited state and the field has exactly n photons show that the probability for finding the atom in the excited state and the field with n photons at a time t is

$$P_2(t) = |\langle n,1| e^{-iH_1 t/\hbar} |n,1\rangle|^2 = \cos^2 \Omega t. \tag{21}$$

This is the Rabi nutation of the atom with Ω being the Rabi frequency which was already found in the semi-classical theory of electromagnetism.

e) Consider now a light field in a coherent state (Glauber state) coupled to the atom in the excited state, show that the probability of finding the atom in the excited state after a time t is

$$P_2(t) = \frac{1}{2}\left[1 + \sum \frac{e^{-\bar{n}}\bar{n}^n}{n!} \cos 2g\sqrt{n+1}t\right] \tag{22}$$

Due to the Poisson distribution of the photon number there is a spread in the Rabi frequencies $\Delta n \sim \bar{n}$. As a result, the Rabi nutation will collapse after some oscillations due to the destructive interference between the various cosine functions. Show that an approximate evaluation of the sum valid for times $t < \bar{n}^{1/2}/g$ yields

$$P_2(t) = \frac{1}{2}\left[1 + \cos 2g(\bar{n}+1)^{1/2} t \exp\left(-\frac{g^2 t^2 \bar{n}}{2(\bar{n}+1)}\right)\right] \tag{23}$$