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Exercise 1: Klein-Gordon theory for the hydrogen atom (Written, 4pts)

The *non*-relativistic (stationary) Schrödinger equation for the hydrogen atom can be written as (the radial part)

$$\left[-\left(\partial_r^2 + \frac{2}{r}\partial_r\right) + \frac{l(l+1)}{r^2} - \frac{2mcZ\alpha}{\hbar r} - E\frac{2m}{\hbar^2}\right]\psi(r) = 0 \tag{1}$$

with the fine-structure constant $\alpha = e^2/\hbar c$. On the other hand, the (stationary) Klein-Gordon equation takes the following form:

$$\left[c^2\hbar^2\Delta + \left(E + \frac{Ze^2}{r}\right)^2 - m^2c^4\right]\psi(r) = 0 \tag{2}$$

Show that (1) can be transformed into (2) by using the following substitutions:

$$\begin{aligned} l(l+1) &\longrightarrow l(l+1) - Z^2\alpha^2 = \lambda(\lambda+1), \\ E &\longrightarrow \frac{E^2 - m^2c^4}{2mc^2}, \\ \alpha &\longrightarrow \alpha\frac{E}{mc^2}. \end{aligned} \tag{3}$$

Here, we have defined $\lambda = l - \delta_l$ and the quantum defect

$$\delta_l = l + 1/2 - \sqrt{(l+1/2)^2 - Z^2\alpha^2}. \tag{4}$$

Consequently, we can find the energies of the Klein-Gordon problem by using the analogy to the non-relativistic case . Show that the energies are given by

$$E_{n,l} = mc^2\left[1 + \left(\frac{Z\alpha}{n - \delta_l}\right)^2\right]^{-1/2}. \tag{5}$$

Note: Use the hydrogen atom energies which include the quantum defect δ_l. Compare the solution to the energies that one gets from the Dirac equation:

$$E_{n,j} = mc^2\left[1 + \left(\frac{Z\alpha}{n - (j+1/2) + \sqrt{(j+1/2)^2 - (Z\alpha)^2}}\right)^2\right]^{-1/2}. \tag{6}$$

What is missing in the Klein-Gordon description of the hydrogen atom?

Exercise 2: Symmetries of the Schrödinger equation (Oral)

In the first part of this exercise, we show that the free Schrödinger equation is invariant under a Galilei transformation. In the second part, we prove the invariance under a gauge transformation (Eichtransformatione).

a) Consider two reference frames F and F' with coordinates (x, t) and (x', t'), respectively. The frame F' moves relatively to frame F with constant velocity v, such that

$$x = x' + vt', \quad t = t'. \tag{7}$$

The (1D) Schrödinger equation for a free particle of mass m in frame F is given by

$$i\hbar\partial_t \Psi(x,t) = -\frac{\hbar^2}{2m}\partial_x^2 \Psi(x,t). \tag{8}$$

The Galilei transformation from $F \longrightarrow F'$ changes the wave function:

$$\Psi(x,t) \longrightarrow \Psi'(x',t') = \mathrm{e}^{if(x',t')}\,\Psi(x'+vt',t'). \tag{9}$$

where f is a real-valued function that depends on space and time. Determine the function f such that $\Psi'(x', t')$ fulfills the Schrödinger equation in frame F', i.e. show that the Schrödinger equation is invariant under Galilei transformations.

b) Let us now examine the behavior of the (3D) Schrödinger equation under gauge transformations. For a massive particle of charge q, we have

$$\left[\frac{1}{2m}\left(\mathbf{p} - \frac{q}{c}\,\mathbf{A}\right)^2 + q\,\phi\right]\Psi = i\hbar\,\partial_t\Psi\,, \tag{10}$$

where $\mathbf{A}$ is the vector potential and ϕ is the scalar potential. We apply a gauge transformation

$$\begin{aligned} \mathbf{A} &\longrightarrow \mathbf{A}' = \mathbf{A} + \nabla\chi\,, \\ \phi &\longrightarrow \phi' = \phi - \frac{1}{c}\partial_t\chi\,, \end{aligned} \tag{11}$$

where χ is a scalar-valued function. In order for the Schrödinger equation to be gauge invariant, we want

$$\left[\frac{1}{2m}\left(\mathbf{p} - \frac{q}{c}\,\mathbf{A}'\right)^2 + q\,\phi'\right]\Psi' = i\hbar\,\partial_t\Psi'\,. \tag{12}$$

to be fulfilled. How do you have to choose Ψ' in order for this equation to be consistent with equation (10)? Use the same technique as in the first part and show that both equations are equivalent.