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This exercise sheet is concerned with relativistic electrons in a magnetic field. We are going to solve the Dirac equation in the presence of a homogeneous magnetic field and determine the eigenenergies, their degeneracy as well as the wave functions.

Remark: In the following, $\ker H$ refers to the kernel of the operator H, i.e. the subspace of the Hilbert space which is mapped to $\mathbf{0}$. $(\ker H)^{\perp}$ denotes its orthogonal supplement, i.e. the subspace which is orthogonal to $\ker H$.

Exercise 1: Relativistic electrons in a magnetic field, part I (Oral)

We start with the derivation of a necessary condition for the eigenvalues of Dirac operators of the form

$$H = \sum_{i=1}^{3} \alpha_i \pi_i + m\beta, \tag{1}$$

where $\pi_i = p_i - \frac{e}{c} A_i$ is the canonical momentum of a particle with charge e in a magnetic field with vector potential $\mathbf{A}$. Calculate H^2 and derive an equation for the eigenvalues. Which non-relativistic problem can be used to determine the eigenvalues?

Exercise 2: Foldy-Wouthuysen Transformation (Oral)

We have already encountered the Foldy-Wouthuysen transformation in the lecture for the derivation of the non-relativistic limit. It is generally applicable for a problem of the form $H = Q + m\beta$, where Q and β are given by

$$Q = \begin{pmatrix} 0 & D^\dagger \\ D & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}. \tag{2}$$

Because Q and H are self-adjoint operators, they have a polar decomposition, for example $H = |H| \operatorname{sgn} H = \operatorname{sgn} H |H|$ with $|H| = \sqrt{H^2}$ and $\operatorname{sgn} H \equiv |H|^{-1} H$ on $(\ker H)^{\perp}$ and $\operatorname{sgn} H = 0$ on $\ker H$. Show that

$$U_{FW} = a_+ + \beta(\operatorname{sgn} Q) a_-, \qquad \text{with } a_\pm = \frac{1}{\sqrt{2}} \sqrt{1 \pm m|H|^{-1}} \tag{3}$$

is a unitary operator which diagonalizes H, i.e. show that

$$H_{FW} = U_{FW} H U_{FW}^\dagger = \beta |H| = \begin{pmatrix} \sqrt{D^\dagger D + m^2} & 0 \\ 0 & -\sqrt{D D^\dagger + m^2} \end{pmatrix}. \tag{4}$$

Also, show that $H_{FW}^2 = H^2$.

Exercise 3: Relativistic electrons in a magnetic field, part II (Written, 4 points)

With the help of the Foldy-Wouthuysen transformation from exercise 2, we can then determine the spectrum of the Dirac-Equation in a magnetic field in two dimensions. In order to do that, we set $\pi_3 = 0$ and constrain the wave functions to the xy plane.

a) Show that the Dirac-Equation separates into two (two-dimensional) spinor spaces with a two-by-two Hamiltonian of the analogous form $H = Q + m\beta$.

b) Determine the spectrum and the eigenfunction of the operators $D^\dagger D$ and $DD^\dagger$.

   The ground state of $N = D^\dagger D$ satisfies the equation $D\psi_0 = 0$, which can be solved with the ansatz $\psi_0 = e^{-\phi}\omega$ where $\phi = eB/4 \cdot (x_1^2 + x_2^2)$.

c) Show that the ground state has a high degeneracy. Introduce an additional quantum number for the z-component of the angular momentum $J_3 = -ix_1\partial_2 + ix_2\partial_1 + \sigma_3/2$ (the angular momentum operator has the same form after the F-W-transformation).

d) The operators D and $D^\dagger$ satisfy $[D, D^\dagger] = 2eB$. Consequently, they can be regarded as ascending and descending operators. Use this to determine the spectrum of the Dirac-equation.

Exercise 4: Relativistic electrons in a magnetic field, part III: Supersymmetry (Written, Bonus, 4 points)

The Dirac-Equation is an example for a concept called supersymmetry. Note that β is an involution (a self-inverse mapping), i.e. $\beta^2 = 1$. The Dirac operator $H = Q + m\beta$ is given in terms of an *odd* operator Q which anticommutes with β and an *even* part, $m\beta$.

The so-called supercharge $Q = Q^\dagger$ is of the form (2). As the F-W transformation shows, we can determine its properties from the supersymmetric Hamiltonian Q^2. Use the polar decomposition of Q to find an isometry between $(\ker D)^\perp$ and $(\ker D^\dagger)^\perp$, in order to show the unitary equivalence of $DD^\dagger$ and $D^\dagger D$. Use $[D, D^\dagger] = 2eB$ to derive the energy gaps in the spectrum of $DD^\dagger$ and $D^\dagger D$.