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Exercise 1: Quantum computing: Fourier transform (Oral)

a) Consider an N-dimensional Hilbert space with $N = 2^n$ states $|j\rangle$, where the index j runs from 0 to $2^n - 1$. We define the transformation U via

$$U|j\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \mathrm{e}^{2\pi i jk/N} |k\rangle . \tag{1}$$

Show that the operator U is unitary.

b) Let $|\psi\rangle = \sum_{j=0}^{N-1} x_j |j\rangle$ be a general state. Show that the amplitudes $y_k = \langle k|U|\psi\rangle$ are just the discrete Fourier transform of the amplitudes x_j.

c) We consider the case $n = 2$, i.e. $N = 4$. The four-dimensional Hilbert-space can be spanned by the product states $|s_1, s_2\rangle$, where the indices $s_i \in \{0, 1\}$ label the states of two qubits. The mapping between the basis $|s_1, s_2\rangle$ and the basis $|j\rangle$ is given by the binary representation of the number $j = 2s_1 + s_2$.

Show that the application of U can be written as

$$U|j\rangle = U|s_1, s_2\rangle = \frac{1}{2}\Big(|0\rangle + \mathrm{e}^{2\pi i s_2/2}|1\rangle\Big)\Big(|0\rangle + \mathrm{e}^{2\pi i(s_1/2+s_2/4)}|1\rangle\Big) . \tag{2}$$

Given this decomposition, we can see that U can be represented by single-qubit and two-qubit gates.

d) Start with the state $|j\rangle$ and apply the Hadamard gate (basis: $|0\rangle, |1\rangle$):

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{3}$$

to the first qubit. Show that this operation creates the state

$$\frac{1}{\sqrt{2}}\Big(|0\rangle + \mathrm{e}^{2\pi i s_1/2}|1\rangle\Big)|s_2\rangle . \tag{4}$$

e) We would now like to add a phase-shift of the form $\mathrm{e}^{2\pi i s_2/4}$, in order to create the state

$$\frac{1}{\sqrt{2}}\Big(|0\rangle + \mathrm{e}^{2\pi i(s_1/2+s_2/4)}|1\rangle\Big)|s_2\rangle . \tag{5}$$

Which two-qubit gate creates this unitary transformation?

f) Finally, a Hadamard gate applied to the second qubit and a SWAP operation to exchange the two qubits can be used to complete the transformation U. How does U_{SWAP} look like?

g) Generalize this pattern to an arbitrary number n and show that the number of elementary operations (number of single- and two-qubit gates) scales like n^2. Compare this to the number of elementary arithmetic operations that a classical computer needs to compute the discrete Fourier transform.

Note: There is no known way to efficiently measure the *amplitudes* y_k such that this would result in a speed-up over the classical Fourier transform. However, the quantum Fourier transform is used as a crucial step in many quantum algorithms.

Exercise 2: Liouville–von Neumann equation (Oral)

Let ρ be a density matrix given by $\rho = \sum_i p_i \left|\psi_i\right\rangle\left\langle\psi_i\right|$. Show that the time evolution of the density matrix is determined by the equation:

$$\partial_t \rho = -\frac{i}{\hbar}\,[H, \rho] \qquad \text{(Liouville-von Neumann equation).} \tag{6}$$

Exercise 3: Partial trace (Oral)

Consider a quantum system S with Hilbert space $\mathcal{H}_S = \mathcal{H}_A \otimes \mathcal{H}_B$, which consists of two subsystems A and B. Let $\{\left|i\right\rangle_A\}$ and $\{\left|k\right\rangle_B\}$ be sets of orthonormal basis states in the respective Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$.

Note that each pure state $\left|\psi\right\rangle \in \mathcal{H}_S$ of the full system can be written as

$$\left|\psi\right\rangle = \sum_{i,k} c_{i,k} \left|i\right\rangle_A \otimes \left|k\right\rangle_B\,, \qquad \sum_{i,k} |c_{i,k}|^2 = 1. \tag{7}$$

Let $L(\mathcal{H}_i)$ be the space of linear operators over the Hilbert space $\mathcal{H}_i$. We define the *partial trace* (Partialspur) with respect to subsystem B:

$$\begin{aligned} \mathrm{Tr}_B : L(\mathcal{H}_S) &\longrightarrow L(\mathcal{H}_A)\,, \\ \mathrm{Tr}_B\, X &= \sum_{m\in B} \left\langle m\right|_B X \left|m\right\rangle_B\,. \end{aligned} \tag{8}$$

a) Let $M_A = M'_A \otimes \mathbb{1}_B \in L(\mathcal{H}_S)$ be an operator which acts only on subsystem A (and trivially on subsystem B). Show that the expectation value of M_A in the state $\left|\psi\right\rangle$ can be written as

$$\langle\psi|M_A|\psi\rangle = \mathrm{Tr}[M_A \cdot \rho] = \mathrm{Tr}[M'_A \cdot \rho_A] \tag{9}$$

where $\rho_A = \mathrm{Tr}_B\, \rho = \mathrm{Tr}_B \left|\psi\right\rangle\left\langle\psi\right|$ is the so-called *reduced density matrix.* Note that the right hand side is computed entirely in subsystem A.

b) As an example, consider a system which consists of two spin-1/2 subsystems. Let the full system be in the state

$$\left|\psi\right\rangle = u \left|\uparrow\right\rangle_A \left|\uparrow\right\rangle_B + v \left|\downarrow\right\rangle_A \left|\downarrow\right\rangle_B\,. \tag{10}$$

Determine the reduced density matrix ρ_A. Does ρ_A describe a pure state or a mixed state?