Exercise 1: Quantum computing: Fourier transform (Oral)

a) Consider an $N$-dimensional Hilbert space with $N = 2^n$ states $|j\rangle$, where the index $j$ runs from $0$ to $2^n - 1$. We define the transformation $U$ via

$$U |j\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi i j k/N} |k\rangle.$$  

(1)

Show that the operator $U$ is unitary.

b) Let $|\psi\rangle = \sum_{j=0}^{N-1} x_j |j\rangle$ be a general state. Show that the amplitudes $y_k = \langle k|U|\psi\rangle$ are just the discrete Fourier transform of the amplitudes $x_j$.

c) We consider the case $n = 2$, i.e. $N = 4$. The four-dimensional Hilbert-space can be spanned by the product states $|s_1, s_2\rangle$, where the indices $s_i \in \{0, 1\}$ label the states of two qubits. The mapping between the basis $|s_1, s_2\rangle$ and the basis $|j\rangle$ is given by the binary representation of the number $j = 2s_1 + s_2$.

Show that the application of $U$ can be written as

$$U |j\rangle = U |s_1, s_2\rangle = \frac{1}{2} \left( |0\rangle + e^{2\pi i s_2/2} |1\rangle \right) \left( |0\rangle + e^{2\pi i (s_1/2 + s_2/4)} |1\rangle \right).$$  

(2)

Given this decomposition, we can see that $U$ can be represented by single-qubit and two-qubit gates.

d) Start with the state $|j\rangle$ and apply the Hadamard gate (basis: $|0\rangle, |1\rangle$):

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$  

(3)

to the first qubit. Show that this operation creates the state

$$\frac{1}{\sqrt{2}} \left( |0\rangle + e^{2\pi i s_1/2} |1\rangle \right) |s_2\rangle.$$  

(4)

e) We would now like to add a phase-shift of the form $e^{2\pi i s_2/4}$, in order to create the state

$$\frac{1}{\sqrt{2}} \left( |0\rangle + e^{2\pi i (s_1/2 + s_2/4)} |1\rangle \right) |s_2\rangle.$$  

(5)

Which two-qubit gate creates this unitary transformation?

f) Finally, a Hadamard gate applied to the second qubit and a SWAP operation to exchange the two qubits can be used to complete the transformation $U$. How does $U_{\text{SWAP}}$ look like?
g) Generalize this pattern to an arbitrary number $n$ and show that the number of elementary operations (number of single- and two-qubit gates) scales like $n^2$. Compare this to the number of elementary arithmetic operations that a classical computer needs to compute the discrete Fourier transform.

**Note:** There is no known way to efficiently measure the amplitudes $y_k$ such that this would result in a speed-up over the classical Fourier transform. However, the quantum Fourier transform is used as a crucial step in many quantum algorithms.

**Exercise 2: Liouville–von Neumann equation (Oral)**

Let $\rho$ be a density matrix given by $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$. Show that the time evolution of the density matrix is determined by the equation:

$$\partial_t \rho = -\frac{i}{\hbar} [H, \rho] \quad \text{(Liouville-von Neumann equation)}. \quad (6)$$

**Exercise 3: Partial trace (Oral)**

Consider a quantum system $S$ with Hilbert space $\mathcal{H}_S = \mathcal{H}_A \otimes \mathcal{H}_B$, which consists of two subsystems $A$ and $B$. Let $\{|i\rangle_A\}$ and $\{|k\rangle_B\}$ be sets of orthonormal basis states in the respective Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$.

Note that each pure state $|\psi\rangle \in \mathcal{H}_S$ of the full system can be written as

$$|\psi\rangle = \sum_{i,k} c_{i,k} |i\rangle_A \otimes |k\rangle_B, \quad \sum_{i,k} |c_{i,k}|^2 = 1. \quad (7)$$

Let $L(\mathcal{H}_i)$ be the space of linear operators over the Hilbert space $\mathcal{H}_i$. We define the partial trace (Partialspur) with respect to subsystem $B$:

$$\text{Tr}_B : L(\mathcal{H}_S) \longrightarrow L(\mathcal{H}_A), \quad \text{Tr}_B X = \sum_{m \in B} \langle m | X | m \rangle_B. \quad (8)$$

a) Let $M_A = M'_A \otimes 1_B \in L(\mathcal{H}_S)$ be an operator which acts only on subsystem $A$ (and trivially on subsystem $B$). Show that the expectation value of $M_A$ in the state $|\psi\rangle$ can be written as

$$\langle \psi | M_A | \psi \rangle = \text{Tr}[M_A \cdot \rho] = \text{Tr}[M'_A \cdot \rho_A] \quad (9)$$

where $\rho_A = \text{Tr}_B \rho = \text{Tr}_B |\psi\rangle\langle\psi|$ is the so-called reduced density matrix. Note that the right hand side is computed entirely in subsystem $A$.

b) As an example, consider a system which consists of two spin-$1/2$ subsystems. Let the full system be in the state

$$|\psi\rangle = u |\uparrow\rangle_A |\uparrow\rangle_B + v |\downarrow\rangle_A |\downarrow\rangle_B. \quad (10)$$

Determine the reduced density matrix $\rho_A$. Does $\rho_A$ describe a pure state or a mixed state?