Exercise 1 - van Hove singularities in one dimension (2 points)

On the last exercise sheet you calculated the dispersion relation for the one-dimensional chain,

$$\omega(q) = \omega_0 |\sin(qa/2)| \ , \tag{1}$$

where ω_0 is the maximum frequency for q at the zone boundary. Show that the density of states (DOS) for this case is

$$N(\omega) = \frac{2}{\pi a \sqrt{\omega_0^2 - \omega^2}} \ , \tag{2}$$

and discuss the van Hove singularities.

Exercise 2 - Creation and annihilation operators (2 points)

Consider the operators $\hat{b}_{\nu q}$ and $\hat{b}^\dagger_{\nu q}$ satisfying the canonical commutation relations for bosons

$$\left[\hat{b}_{\mu q}, \hat{b}^\dagger_{\nu q'}\right] = \delta_{\mu\nu}\delta_{q,q'} \ , \qquad \left[\hat{b}_{\mu q}, \hat{b}_{\nu q'}\right] = \left[\hat{b}^\dagger_{\mu q}, \hat{b}^\dagger_{\nu q'}\right] = 0 \ . \tag{3}$$

The number operator, defined through $\hat{n}_{\nu q} = \hat{b}^\dagger_{\nu q}\hat{b}_{\nu q}$, has eigenstates $|n_{\nu q}\rangle$ with the property

$$\hat{n}_{\nu q} \, |n_{\nu q}\rangle = n_{\nu q} \, |n_{\nu q}\rangle \ , \tag{4}$$

with n some positive integer.

- Show that $\hat{b}_{\nu q}|n_{\nu q}\rangle$ and $\hat{b}^\dagger_{\nu q}|n_{\nu q}\rangle$ are also eigenstates of $\hat{n}_{\nu q}$ with the properties

$$\begin{aligned} \hat{b}_{\nu q}|n_{\nu q}\rangle &= \sqrt{n_{\nu q}} \, |n_{\nu q} - 1\rangle \ , \\ \hat{b}^\dagger_{\nu q}|n_{\nu q}\rangle &= \sqrt{n_{\nu q} + 1} \, |n_{\nu q} + 1\rangle \ . \end{aligned} \tag{5}$$

- Show that there is an eigenstate $|0\rangle$ of $\hat{n}_{\nu q}$ with eigenvalue 0, so that

$$\hat{b}_{\nu q}|0\rangle = 0 \ . \tag{6}$$

  The state $|0\rangle$ in general is called the *vacuum*.

Solutions due on the 29th of April, 2013