Exercise 1 - van Hove singularities in one dimension

On the last exercise sheet you calculated the dispersion relation for the one-dimensional chain,
\[ \omega(q) = \omega_0 | \sin(qa/2) | , \]
where \( \omega_0 \) is the maximum frequency for \( q \) at the zone boundary. Show that the density of states (DOS) for this case is
\[ N(\omega) = \frac{2}{\pi a \sqrt{\omega_0^2 - \omega^2}} , \]
and discuss the van Hove singularities.

Exercise 2 - Creation and annihilation operators

Consider the operators \( \hat{b}_{\nu q} \) and \( \hat{b}^{\dagger}_{\nu q} \) satisfying the canonical commutation relations for bosons
\[ [\hat{b}_{\mu q}, \hat{b}^{\dagger}_{\nu q'}] = \delta_{\mu \nu} \delta_{q,q'} , \quad [\hat{b}_{\mu q}, \hat{b}_{\nu q'}] = [\hat{b}^{\dagger}_{\mu q}, \hat{b}^{\dagger}_{\nu q'}] = 0 . \]
The number operator, defined through \( \hat{n}_{\nu q} = \hat{b}^{\dagger}_{\nu q} \hat{b}_{\nu q} \), has eigenstates \( | n_{\nu q} \rangle \) with the property
\[ \hat{n}_{\nu q} | n_{\nu q} \rangle = n_{\nu q} | n_{\nu q} \rangle , \]
with \( n \) some positive integer.

- Show that \( \hat{b}_{\nu q} | n_{\nu q} \rangle \) and \( \hat{b}^{\dagger}_{\nu q} | n_{\nu q} \rangle \) are also eigenstates of \( \hat{n}_{\nu q} \) with the properties
  \[ \hat{b}_{\nu q} | n_{\nu q} \rangle = \sqrt{n_{\nu q}} | n_{\nu q} - 1 \rangle , \]
  \[ \hat{b}^{\dagger}_{\nu q} | n_{\nu q} \rangle = \sqrt{n_{\nu q} + 1} | n_{\nu q} + 1 \rangle . \]
- Show that there is an eigenstate \( | 0 \rangle \) of \( \hat{n}_{\nu q} \) with eigenvalue 0, so that
  \[ \hat{b}_{\nu q} | 0 \rangle = 0 . \]

The state \( | 0 \rangle \) in general is called the vacuum.

Solutions due on the 29th of April, 2013