Exercise 1 - Wave functions  

Write down the wave functions for the following systems:

(a) Three spinless bosons: two occupying the ground state and one occupying the first excited state.

(b) Three spinless fermions occupying the lowest three states of a quantum system.

Exercise 2 - The ideal Bose gas  

Consider a system of non-interacting Bosons in a three dimensional box with periodic boundary conditions. The Hamiltonian in second quantization reads

\[ H = \sum_{\vec{k}} \epsilon_{\vec{k}} b_{\vec{k}}^\dagger b_{\vec{k}}, \quad \text{where} \quad \epsilon_{\vec{k}} = \frac{\hbar^2 \vec{k}^2}{2m}. \]  

(a) Calculate the density of states

(b) You have seen in the lecture that the partition function for the grand canonical ensemble in case of an ideal Bose gas reads

\[ Z := \text{tr} \left[ e^{-\beta(H - \mu N)} \right] = \prod_{\vec{k}} \frac{1}{1 - \exp \left[ \beta(\mu - \epsilon_{\vec{k}}) \right]}. \]  

Show that the grand potential \( \Omega := -k_B T \ln Z \) can be written as

\[ \Omega(T, V, \mu) = -\frac{2}{3} \frac{V}{4\pi^2} \left( \frac{2m}{\hbar^2} \right)^{\frac{3}{2}} \int_0^\infty d\epsilon \frac{\epsilon^{\frac{3}{2}}}{\exp \left[ \beta(\epsilon - \mu) \right] - 1}. \]  

(c) Show that the equation of states for the ideal Bose gas is given by

\[ pV = \frac{2}{3} E \]  

Remarks:

(1) For homogenous systems we have \( \Omega = -pV \).

(2) Bosons respect Bose-Einstein statistics: the mean occupation number is given by

\[ n(\vec{k}) = \frac{1}{\exp \left[ \beta(\epsilon_{\vec{k}} - \mu) \right] - 1}. \]  

Which values of the chemical potential are physically reasonable?
Exercise 3 - Bose-Einstein condensation (4 points)

We examine again the ideal Bose gas from the last exercise.

(a) Consider the high temperature limit where \( k_B T \gg 1 \) and derive the chemical potential as a function of temperature by considering the mean particle number, given through

\[
N = -\left. \frac{\partial \Omega}{\partial \mu} \right|_{T,V}.
\]

Discuss the behaviour of \( \mu(T) \). Is there a value of the temperature such that \( \mu \) vanishes?

(b) Now we drop the limit of high-\( T \). Show that the temperature at which the chemical potential vanishes is given by:

\[
T_0 \approx 3.31 \frac{\hbar^2}{mk_B} \left( N/V \right)^{\frac{5}{2}},
\]

and verify that the occupation of the lowest energy state diverges in this case.

(c) Show that for \( T \leq T_0 \) the specific heat \( C_V = \frac{\partial E}{\partial T} \mid_{V} \) is given by:

\[
C_V = \frac{\zeta(5/2)\Gamma(5/2)}{\zeta(3/2)\Gamma(3/2)} \cdot 5 \frac{Nk_B}{2} \left( \frac{T}{T_0} \right)^{\frac{5}{2}} \approx 0.77 \cdot \frac{5}{2} Nk_B \left( \frac{T}{T_0} \right)^{\frac{5}{2}}.
\]

(d) For temperatures slightly above \( T_0 \) we approximate the internal energy by:

\[
E(T,V,\mu) \approx E(T \leq T_0, V) + \frac{3}{2} N\mu(T),
\]

and the chemical potential by

\[
\mu(T) \approx -0.54 k_B T_0 \left( \frac{T}{T_0} \right)^{\frac{5}{2}} - 1 \right)^2.
\]

Calculate the specific heat in this regime.

Hint: You might encounter the following integral:

\[
\int_0^\infty dx \frac{x^{a-1}}{e^x - 1} = \Gamma(a)\zeta(a) \quad \text{for } a > 1.
\]

The \( \Gamma \)-function and the Riemann \( \zeta \)-function are well tabulated in the literature or you can compute them with Mathematica.

Solutions due on the 27th of May 2013