Hauptseminar: Physik der kalten Gase

Interaction between atoms

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- scattering theory
  - Schrödinger's equation for scattering:
    $$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR_l}{dr}\right)+\left[k^2-\frac{l(l+1)}{r^2}-\frac{2m}{\hbar^2}\ U(r)\right]R_l=0$$
  - asymptotic solution: $R_l \approx \frac{a_l \cdot \sin(kr-\frac{1}{2}l\pi+\delta_l)}{r}$
  - scattering amplitude: $f_l=\frac{1}{2ik}\left(e^{2i\delta_l}-1\right)$
  - slow particles $\rightarrow$ low energy $\rightarrow$ low temperature
  - phase $\delta_l \sim k^{2l+1}$
  - just care about s-wave scattering $(l=0)$
  - replace interaction potential by pseudo potential
    $V_{\text{eff}}(r)=g\cdot\delta(r)(\partial_r\ r)$
  - s-wave scattering length $a_s=\frac{1}{4\pi}\frac{m}{\hbar^2}g$
  - $\rightarrow$ use perturbation theory for scattering

- Hamiltonian for interaction:

$$\hat{H}=\int dx\hat{\Psi}^\dagger(x)\ (-\frac{\hbar^2}{2m}\nabla^2)\ \hat{\Psi}(x)$$

$$+\frac{1}{2}\int\hat{\Psi}^\dagger(x)\hat{\Psi}^\dagger(x')\ V(x'-x)\hat{\Psi}(x)\hat{\Psi}(x')dxdx'$$

- equation of motion: $i\hbar\frac{\partial}{\partial t}\hat{\Psi}(\mathbf{r},t)=\left[\hat{\Psi}(\mathbf{r},t),\hat{H}\right]$

- Gross-Pitaevskii:
  - macroscopic occupation of one mode
  - pseudo potential & lowest order approximation
  - Gross-Pitaevskii equation:
    $$i\hbar\frac{\partial}{\partial t}\Psi_0(\mathbf{r},t)=\left(-\frac{\hbar^2\nabla^2}{2m}+V_{ext}(\mathbf{r},t)+g|\Psi_0(\mathbf{r},t)|^2\right)\Psi_0(\mathbf{r},t)$$
  - describes interacting gas in external field

- Bogoliubov theory
  - Hamiltonian:
    $$\hat{H} = \sum_{\mathbf{p}} \frac{p^2}{2m} \hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{p}} + \frac{1}{2V} \sum_{\mathbf{p_1},\mathbf{p_2},\mathbf{q}} V_{\mathbf{q}} \hat{a}^\dagger_{\mathbf{p_1}+\mathbf{q}} \hat{a}^\dagger_{\mathbf{p_2}-\mathbf{q}} \hat{a}_{\mathbf{p_1}} \hat{a}_{\mathbf{p_2}}$$

- Bogoliubov lowest order approximation:
  - neglect all particles with $\mathbf{p} \neq 0$
  - pseudo potential with first Born approximation $V_0 = g = \frac{4\pi\hbar^2 a}{m}$
  - ground state energy: $E_0 = \frac{N^2}{2V} g$
  - compressibility: $\frac{\partial n}{\partial P} = \frac{1}{gn} = \frac{1}{mc^2}$
  - $\rightarrow$ sound velocity: $c = \sqrt{\frac{gn}{m}}$

- Higher order approximation
  - neglect quadratic terms of $\mathbf{p} \neq 0$
  - pseudo potential with higher order: $V_0 = g\left(1 + \frac{g}{V}\sum_{\mathbf{p}\neq 0} \frac{m}{p^2}\right)$
  - Hamiltonian before transformation:

$$\hat{H} = g\frac{N^2}{2V} + \sum \frac{p^2}{2m} \hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{p}} + \frac{1}{2} gn \sum_{\mathbf{p}\neq 0} \left(2\hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{p}} + \hat{a}^\dagger_{\mathbf{p}} \hat{a}^\dagger_{-\mathbf{p}} + \hat{a}_{\mathbf{p}} \hat{a}_{-\mathbf{p}} + \frac{mgn}{p^2}\right)$$

  - Hamiltonian after transformation:
    $$\hat{H} = E_0 + \sum \epsilon(p) \hat{b}^\dagger_{\mathbf{p}} \hat{b}_{\mathbf{p}}$$
  - ground state energy: $E_0 = g\frac{N^2}{2V}\left[1 + \frac{128}{15\sqrt{\pi}}(na^3)^{1/2}\right]$
  - dispersion law: $\epsilon(p) = \left[\frac{gn}{m}p^2 + \left(\frac{p^2}{2m}\right)^2\right]^{1/2}$
  - characteristic healing length: $\xi = \sqrt{\frac{\hbar}{2mgn}} = \frac{1}{\sqrt{2}}\frac{\hbar}{mc}$
  - occupation quasi particles: $N_{\mathbf{p}} \equiv \langle \hat{b}^\dagger_{\mathbf{p}} \hat{b}_{\mathbf{p}} \rangle = \frac{1}{\exp[\beta\epsilon(p)]-1}$
  - occupation particles:
    $$n_{\mathbf{p}} = \langle \hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{p}} \rangle = |v_{-\mathbf{p}}|^2 + |u_{\mathbf{p}}|^2 \langle \hat{b}^\dagger_{\mathbf{p}} \hat{b}_{\mathbf{p}} \rangle + |v_{-\mathbf{p}}|^2 \langle \hat{b}^\dagger_{-\mathbf{p}} \hat{b}_{-\mathbf{p}} \rangle$$
  - number of particles in the condensate at $T = 0K$:
    $$N_0 = N - \sum_{\mathbf{p}\neq 0} n_{\mathbf{p}}$$
  - $\rightarrow n_0 \equiv \frac{N_0}{V} = n\left[1 - \frac{8}{3\sqrt{\pi}}(na^3)^{1/2}\right]$